$f : A \to B$ $A, B \in \mathcal{C}$ category

kernel of f ,

$$\begin{tikzcd} K \arrow[r] & A \arrow[r] & B \\ \forall D & \exists ! & D \arrow[ull, "\exists !"] \arrow[ul] \arrow[u] \arrow[ur, phantom] & \end{tikzcd}$$

$$\begin{tikzcd} K \arrow[r] & A \arrow[r] & B \\ & D \arrow[ul, dashed, "\exists !"] \arrow[u] \arrow[ur, "0"'] & \end{tikzcd}$$

kernel is a monomorphism.

cokernel of f

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "0"'] & B \arrow[r] \arrow[d] & C \arrow[dl, dashed, "\exists !"] & \forall C \\ \forall D & D & & \end{tikzcd}$$

coker is epi

$\oplus$

Idea of abelian category

Behaves like ab. gps, modules over comm ring, etc

Examples

- Ab — abelian groups
- Mod(A) — mod. over comm ring A
- Ab(X) — sheaves of ab. grps on top space X
- Mod(X) — $(X, \mathcal{O}_X)$ ringed space — sheaves of $\mathcal{O}_X$-mod.
- QCo(X) — X scheme : quasicoherent sheaves
- Coh(X) — X Noeth sch — coherent sheaves

Def An abelian category $\mathcal{C}$ $\forall A, B \in \mathcal{C}$

- $\mathrm{Hom}(A,B)$ ab. gp
- $\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$ comp. bilin.
- $\oplus$ exist
- Every morph. has a kernel + cokernel
- Every monomorphism is kernel of its coker.
- Every epimorphism coker of its ker

$$A \xrightarrow{f} B \xrightarrow{g} \mathrm{coke}\, f$$

$\ker g = \mathrm{Im}\, A \cong A$

if f mono-

$\mathcal{C}$ abelian category

$C(\mathcal{C})$ of chain complexes in $\mathcal{C}$

$A^\bullet \in C(\mathcal{C})$ $\quad \cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^{i} \xrightarrow{d^{i}} A^{i+1} \to \cdots$

$d^i \circ d^{i-1} = 0$

$h^i(A^\bullet) \in \mathcal{C}$

$$h^i(A^\bullet) = \frac{\ker d^i}{\operatorname{im} d^{i-1}} \in \mathcal{C}$$

$f : A^{\cdot} \longrightarrow B^{\cdot} \quad \in C^{\cdot}(\mathcal{A})$

$$\begin{tikzcd}
\arrow[r] & A^{i} \arrow[r, "d_A^{i}"] \arrow[d, "f^{i}"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & {} \\
\arrow[r] & B^{i} \arrow[r, "d^{i}"] & B^{i+1} &
\end{tikzcd}$$

$f^{i+1} d^{i} = d^{i} f^{i}$

$h^{i}(f) : h^{i}(A^{\cdot}) \longrightarrow h^{i}(B^{\cdot})$

$$\left. \begin{array}{l} \ker d_A^{i} \xrightarrow{f^{i}} \ker d_B^{i} \\ \operatorname{Im} d_A^{i-1} \xrightarrow{f^{i}} \operatorname{Im} d_B^{i-1} \end{array} \right|$$

induce $h^{i}(f)$ on $\dfrac{\ker d_A^{i}}{\operatorname{Im} d_A^{i}}$

Short exact sequence of complexes

$0 \to A^{i} \to B^{i} \to C^{i} \to 0$

s.e.s.

$0 \longrightarrow A^{\cdot} \longrightarrow B^{\cdot} \longrightarrow C^{\cdot} \longrightarrow 0$

$\delta^{i} : h^{i}(C^{\cdot}) \longrightarrow h^{i+1}(A^{\cdot})$

$\delta^i : H^i(C^\cdot) \to H^{i+1}(A^\cdot)$

$\qquad\quad \rotatebox{90}{$\in$} \qquad\qquad\quad \rotatebox{90}{$\in$}$

$g \in \ker d^i \quad [g] \mapsto \underset{\thicksim}{[e]}$

$e \in \ker d_A^{i+1}$

$$\begin{tikzcd}
 & B^i \arrow[r] & C^i \arrow[r] & 0 \\
 & f \arrow[r, mapsto] \arrow[d, "d_B^i"'] & g \arrow[r] \arrow[d, "d^i"] & 0 \\
0 \arrow[r] & e \arrow[r] \arrow[d] & df \arrow[r] \arrow[d, phantom, sloped, "\in"] & 0 \arrow[r] & 0 \\
 & A^{i+1} \arrow[d] & B^{i+1} \arrow[d] & & \\
 & 0=de \arrow[r] & d^2f=0 & &
\end{tikzcd}$$

Ex: δ^i indep of choice of f

Prop $0 \to A^\cdot \to B^\cdot \to C^\cdot \to 0$ ses

gives rise to long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & h^i(A^\cdot) \arrow[r] & h^i(B^\cdot) \arrow[r] & h^i(C^\cdot) \arrow[dll, "\delta^i"] \\
 & h^{i+1}(A^\cdot) \arrow[r] & h^{i+1}(B^\cdot) \arrow[r] & h^{i+1}(C^\cdot) \arrow[dll, "\delta^{i+1}"] \\
 & h^{i+1}(A^\cdot) \arrow[r] & \cdots &
\end{tikzcd}$$

Pf ① Can be proven directly from the axioms for abelian categories (hard!)

② Use Freyd embedding theorem and diagram chasing

$\mathcal{A} \subset \mathrm{Mod}(A)$ some A

Homotopy

$f, g : A^\cdot \longrightarrow B^\cdot$

$f \sim g$

$$\begin{tikzcd}
{} \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] \arrow[dl, "k^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] \arrow[dl, "k^{i+1}"'] & {} \\
& B^{i-1} \arrow[r, "d^{i-1}"'] & B^i \arrow[r, "d^i"'] & B^{i+1} \arrow[r] & {}
\end{tikzcd}$$

$$f^i - g^i = d^{i-1} k^i + k^{i+1} d^i$$

Prop: If $f \sim g$, then $h^i(f) = h^i(g)$

Pf: $h^i(A^\cdot) \ni [h] \qquad h \in \ker d^i_A$

$h^i(f) - h^i(g)$ induced by $d^{i-1} k^i + k^{i+1} d^i$

Apply to h $\quad \left(\underbrace{d^{i-1} k^i h}_{\in \operatorname{Im} d^{i-1}} + k^{i+1} \underbrace{d^i h}_{0}\right) =$

$[d^{i-1} k^i h] = 0$ in $h^i(B^\cdot)$

$h^i(f) = h^i(g)$

$\mathcal{C}, \mathcal{D}$ abelian categories $F : \mathcal{C} \to \mathcal{D}$ covariant functor

Def F is **additive** if $\forall A, B \in \mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

is a homom. of ab. grps

F is **left exact** if it is additive and for all ses $0 \to A \to B \to C \to 0$ ses

$$0 \to F(A) \to F(B) \to F(C)$$ exact

right exact $F(A) \to F(B) \to F(C) \to 0$ exact

exact $0 \to F(A) \to F(B) \to F(C) \to 0$ exact

For contravariant functors

$0 \to F(C) \to F(B) \to F(A)$ exact (left exact)

$F(C) \to F(B) \to F(A) \to 0$ right exact

Ex $\mathrm{Hom}(\cdot, A) : \mathcal{C}^{o} \to Ab$ left exact

$\mathrm{Hom}(A, \cdot)$ $\mathcal{C} \to Ab$ left exact

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"] & B \arrow[dl, "\exists ?"] \\ I & \end{tikzcd}$$

$\mathcal{C}$ abelian category

Def $I \in \mathcal{C}$ injective if

$\mathrm{Hom}(-, I)$ exact

$0 \to A \to B$

$\mathrm{Hom}(B, I) \to \overset{f}{\mathrm{Hom}(A, I)} \to 0$

Def $A \in \mathcal{C}$ injective resolution of A

$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$ I^j inject

exact

Ex: $\mathcal{C} = Ab$

Def: $G \in Ab$ is divisible if $\forall g \in G$ $\forall n \in \mathbb{Z}_+$

$\exists h \in G$ s.t. $g = nh$

Thm G is injective $\Longleftrightarrow$ G is divisible

Pf Suppose G not divisible.
Pick $g \in G$, $n \in \mathbb{Z}$ s.t. $\nexists\, h \in G$
$g = nh$

$$\begin{tikzcd}
G & \mathbb{Z} \arrow[l, hook'] \arrow[dr] & \subset & \mathbb{Z}(\frac{1}{n}) \arrow[dl, dashed, "\exists ?"'] & \subset & \mathbb{Q} \\
& & G & & &
\end{tikzcd}$$

$1 \mapsto g$ $\qquad$ $h \mapsfrom \frac{1}{n}$

$1 = n(\frac{1}{n})$

$\therefore\ g = nh$

So g not divis.

Suppose G divisible, show G injective

$$\begin{tikzcd}
A \arrow[dr, "\alpha"'] & \subset & B \arrow[dl, "\exists ?"] \\
& G &
\end{tikzcd}$$

By a Zorn's lemma argument, suffices to show $\exists\, H$,

$$\begin{tikzcd}
A \arrow[dr] & \subsetneq & H \arrow[dl] & \subset & B \\
& G & & &
\end{tikzcd}$$

Will show existence of H

Pick $x \in B - A$

① Suppose $nx \notin A$ $\forall n \in \mathbb{Z}_+$.

Then

$$\begin{tikzcd}
A \arrow[r, phantom, "\subsetneq"] \arrow[dr, "\varphi"'] & H = A + \mathbb{Z}x \subset B \arrow[d, "\psi"] \\
& G
\end{tikzcd}$$

$\psi(a + nx) = \varphi(a)$

② Suppose $nx \in A$, $n \in \mathbb{Z}_+$ minimal

Pick $g \in G$ $ng = \varphi(nx)$

$$\begin{tikzcd}
A \oplus \mathbb{Z} \arrow[rr, "\rho"] \arrow[d, "\varphi'"'] \arrow[dr] & & G \\
B\,? & \operatorname{Im}\varphi' \arrow[ur, "\exists"'] & \\
& H \arrow[u, phantom, sloped, "="] &
\end{tikzcd}$$

$\rho(a, m) = \varphi(a) + mg$

$\ker \varphi' \subset \ker \rho$

$\varphi'(a, m) = \varphi(a) + mx$

Prop $G \in Ab$. Then $\exists$ injective H,
$G \subset H$

Pf (sketch)

$\mathbb{Z} \subset \mathbb{Q}$

$\mathbb{Z}/n\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$ $\qquad 1 \mapsto \frac{1}{n}$

$g \in G - 0$

$$\begin{tikzcd} \mathbb{Z}g \arrow[r, phantom, "\subset"] \arrow[d, "i_g"'] & G \arrow[dl, dashed, "f_g"] & \ni g \arrow[d] \\ I_g := \mathbb{Q} \text{ or } \mathbb{Q}/\mathbb{Z} & & \neq 0 \end{tikzcd}$$

$$G \xhookrightarrow{\prod f_g} \prod_{g \in G - 0} I_g \quad \text{inj.}$$

$\mathcal{C}$ abelian category

Def $\mathcal{C}$ has enough injectives

$\forall A \in \mathcal{C}$ $\exists$ inj $I \in \mathcal{C}$ $A \hookrightarrow I$

Examples Ab, $Mod(A)$, $Ab(X)$, $Mod(X)$, $QCoh(X)$, $Coh(X)$

Lemma $\mathcal{C}$ has enough injectives $\Rightarrow$ every $A \in \mathcal{C}$ has an inj res.

Pf $0 \to A \xrightarrow{\alpha} I^0 \longrightarrow I^1 \longrightarrow \cdots$

$I^0 \longrightarrow \underline{I^0/\mathrm{Im}\,\alpha} \underline{\hookrightarrow} I^1$

Lemma: $A \to I^\cdot$, $A \to J^\cdot$ two injective resolutions. Then $I^\cdot$ and $J^\cdot$ are homotopy equivalent.

Pf (sketch)

$I^0/\operatorname{Im} \alpha$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, equal] & I^0 \arrow[r] \arrow[d] \arrow[rd, bend right] & I^1 \arrow[r] \arrow[d, shift right] & I^2 \arrow[d, shift right, "g"'] \\
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] \arrow[u, bend left] & J^1 \arrow[r] \arrow[u, shift right, "f"'] & J^2 \arrow[u, shift right]
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[r] \arrow[d, "fg"'] & I^1 \arrow[r] \arrow[d, "fg"'] & \cdots \\
0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

hom to 1.

$F : \mathcal{C} \to \mathcal{D}$ covariant left exact

$\forall \; A \in \mathcal{C}$ pick $I^\cdot$ injective res of A

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

$$R^iF(A) := h^i(F(I^\cdot))$$

$$\overset{\text{cx}}{0 \to F(I^0) \xrightarrow{d^0} F(I^1) \xrightarrow{d^1} F(I^2) \to \cdots}$$

$$h^i(F(I^\cdot)) = \ker d^i / \operatorname{Im} d^{i-1} \in \mathcal{D}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I^0 \arrow[d, dashed, shift right] \\
0 \arrow[r] & A \arrow[r] & J \arrow[u, dashed, bend right]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(I^\cdot) \arrow[d, dashed, shift right] \\
F(J^\cdot) \arrow[u, dashed, shift right]
\end{tikzcd}
\qquad
\begin{tikzcd}
h^i(F(I^\cdot)) \arrow[d, shift left, "\cong"] \\
h^i(F(J^\cdot)) \arrow[u, shift left]
\end{tikzcd}$$

diff choices of $I^\cdot$

give isom $R^iF(A)$

$$\boxed{R^0F \cong F}$$

Ex: X top space $F \in Ab(X)$

$$\Gamma(X, \cdot) : Ab(X) \longrightarrow Ab$$

$$H^i(X, F) := R^i\Gamma(X, F) \in Ab$$

Ex $(X, \mathcal{O}_X)$ ringed space (e.g. scheme)

$$\begin{tikzcd} \Gamma(X, \cdot) : \mathrm{Mod}(X) \arrow[r] \arrow[d, hook] & Ab \\ Ab(X) & \end{tikzcd}$$

Prop: derived functors agree

Thm X Noeth top space, $\dim X = n$, $F \in Ab(X)$

then $H^i(X, F) = 0$ $i > n$

Ex X scheme, $\dim X = 0$ Noeth

$$0 \to F' \to F \to F'' \to 0 \quad \text{exact}$$

$$\begin{tikzcd} \Rightarrow & 0 \arrow[r] & \Gamma(X,F') \arrow[r] \arrow[d, phantom, "\parallel"] & \Gamma(X,F) \arrow[r] & \Gamma(X,F'') \arrow[r] & 0 \\ & 0 \arrow[r] & H^0(X,F') \arrow[r] & H^0(F) \arrow[r] & H^0(X,F'') \arrow[dll] & \Gamma \\ & & H^1(X,F'')=0 & & & \end{tikzcd}$$

Ex: X curve $/k$ $F \in Ab(X) \Rightarrow$

$H^i(X,F)=0 \quad \forall\, i \geq 2$

Will see later that if F constant

X irred $\Rightarrow$ $H^i(X,F)=0 \quad \forall\, i \geq 1$

Warning: If $X/\mathbb{C}$: Zariski top $\neq$

Analytic top

Thm $\mathcal{C}$ ab cat with enough injectives, $\mathcal{D}$ abelian

$F : \mathcal{C} \longrightarrow \mathcal{D}$ covariant, left exact. Then

- $R^iF : \mathcal{C} \longrightarrow \mathcal{D}$ addition
- $F \cong R^0F$
- $\forall$ ses $0 \to A' \to A \to A'' \to 0 \in \mathcal{C}$ and $i \geq 0$ have

$$\delta^i : R^iF(A'') \longrightarrow R^{i+1}F(A') \text{ and l.e.s.}$$

$$0 \to R^0F(A') \to R^0F(A) \to R^0F(A'') \xrightarrow{\delta^0} R^1F(A') \to R^1F(A) \to \cdots \quad (\ast)$$

- Given morphism of ses, δ's give

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "h"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "g"] & 0 & = \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 & = \end{tikzcd}$$

$$\begin{tikzcd} R^iF(A'') \arrow[r, "\delta"] \arrow[d, "f"] & R^{i+1}F(A') \arrow[d, "f"] \\ R^iF(B'') \arrow[r, "\delta"] & R^{i+1}F(B') \end{tikzcd}$$

- $I \in \mathcal{C}$ injective, $i > 0 \Longrightarrow$

$$R^iF(I) = 0$$

Pf (some path)

$$0 \to A \to I^0 \xrightarrow{f} I^1 \quad I^0 \to \operatorname{Im} f \hookrightarrow I^1 \quad \operatorname{Im} f \to \cdots$$

$$R^0F(A) = \ker\left(F(I^0) \to F(I^1)\right) = \ker\left(F(I) \to F(\operatorname{Im} f)\right)$$

$$\cong F(A)$$

exact

$$\begin{tikzcd}
0 \arrow[r] & \boxed{F(A)} \arrow[r] & F(I^0) \arrow[r] \arrow[rd, two heads] & F(I^1) \\
& & & F(\operatorname{Im} f) \arrow[u, hook]
\end{tikzcd}$$

exact ?

$$\begin{tikzcd}
& 0 \arrow[d, "\varepsilon'"] & 0 \arrow[d, "\varepsilon"] & 0 \arrow[d, "\varepsilon''"] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (I')^i \arrow[r] & I^i \arrow[r] & (I'')^i \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
A' \arrow[r, hook] \arrow[d, hook] & A \arrow[r] \arrow[d] & A'' \arrow[d] \\
(I^0)' \arrow[r] & I^0 \arrow[r] & (I'')^0 \\
& I^0 \arrow[r] & \boxed{(I^0/I^{0'})} \arrow[u, "?"]
\end{tikzcd}$$

$$0 \to F((I')^i) \to F(I^i) \to F((I'')^i) \to 0 \quad \in D$$

$$R^iF(A') \cong h^i F((I')^i) \to \overset{R^iF(A)}{h^i(F(I'))} \to \overset{R^iF(A'')}{h^i(F(I''))} \to \underset{R^{i+1}F(A')}{h^{i+1} F((I')} \to \cdots$$

i.e.,

$$0 \to I \to \boxed{I \to 0}$$

inj res of I

$$R^iF(I) = h^i\left(F(0 \to \cdots 0 \to I \to 0 \to \cdots)\right)$$

$$= h^i\left(0 \to \cdots 0 \to F(I) \to 0 \cdots\right)$$

$$= \begin{cases} 0 & i \neq 0 \\ F(I) & i = 0 \end{cases}$$

$$\begin{tikzcd}
& 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
\exists & 0 \arrow[r] & (I')^0 \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & (I'')^0 \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & (I')^1 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & (I'')^1 \arrow[r] \arrow[d] & 0 \\
& & \cdots & \cdots & &
\end{tikzcd}$$

rows exact

$$\begin{tikzcd}
0 \arrow[r] & F((I')^0) \arrow[r] \arrow[d] & F(I^0) \arrow[r] \arrow[d] & F(I''^0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F((I')^1) \arrow[r] \arrow[d, dotted] & F(I'^1) \arrow[r] & F(I''^1) \arrow[r] & 0 \\
0 \arrow[r] & \vdots & & &
\end{tikzcd}$$

$R^1F((I')^0) = 0$ exact

i —, 0

Return to $F : \mathcal{A} \to \mathcal{D}$ left exact, $\mathcal{A}$ enough inj-

Def: $J \in \mathcal{A}$ acyclic for F if

$$R^i F(J) = 0 \quad \forall\, i > 0$$

Prop: Suppose $A \in \mathcal{A}$,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & J^0 \arrow[rr] \arrow[dr] & & J^1 \arrow[rr] \arrow[dr] & & J^2 \arrow[r] & {} \\ & & & K^1 \arrow[ur] & & K^2 \arrow[ur] & & \end{tikzcd}$$

J^i acyclic. Then

$$R^i F(A) \simeq h^i(F(J^\bullet)) \in \mathcal{D}$$

Pf (sketch) $0 \to A \to J^0 \to K^1 \to 0$ exact

$0 \to K^1 \to J^1 \to K^2 \to 0$

$0 \to K^i \to J^i \to K^{i+1} \to 0$ exact

$$\underline{R^i F(A)} \simeq \underline{R^{i-1} F(K^1)} \simeq R^{i-2}F(K^2) \simeq \cdots \simeq R^1 F(K^{i-1})$$

$$0 \to F(K^{i-1}) \to F(J^{i-1}) \to F(K^i) \to R^1F(K^{i-1}) \to \underset{\substack{\wr\wr \\ R^1F(J^{i-1})}}{0}$$

$$R^1F(K^{i-1}) \simeq \operatorname{coker}\left(F(J^{i-1}) \to F(K^i)\right) \simeq$$

$$0 \to A \to J^0 \to K^1 \to 0$$

$$\begin{tikzcd}
z^0 \arrow[d] & & & \\
R^{i-1}F(J^0) \arrow[r] \arrow[d, equal] & R^{i-1}F(K^1) \arrow[r, "\delta", "\cong"'] & R^iF(A) \arrow[r] & R^iF(J^0) \arrow[d, phantom, "\shortparallel"] \\
0 & & & (i \geq 2)
\end{tikzcd}$$

$$R^iF(A) \cong \operatorname{coker}\left(F(J^{i-1}) \longrightarrow F(K^i)\right) \overset{\text{Claim}}{\cong} h^iF(J^\bullet)$$

$$F(J^{i-1}) \twoheadrightarrow F(K^i) \twoheadrightarrow F(J^i)$$

$$\cup$$

$$0 \to K^{i-1} \to J^{i-1} \twoheadrightarrow K^i \to 0$$

$$☆\ 0 \to K^i \underline{\to} J^i \to K^{i+1} \to 0$$

$$\left(\ker\left(F(J^i) \underline{\to} F(K^{i+1})\right) \cong \ker F(J^i) \to F(J^{i+1})\right)$$

Derived functors satisfy a univ prop

$\mathcal{C}, \mathcal{D}$ ab cats

Def A (covariant) δ-functor $\mathcal{C} \to \mathcal{D}$

is a collection $\{T^i\}_i : \mathcal{C} \to \mathcal{D}$ and ∀ ses

$0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$ ∃ morph

$$\delta : T^i(A'') \to T^{i+1}(A') \quad \text{s.t.}$$

- LES $0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta} T^1(A') \to \cdots$

- ∀ morphism of ses

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
T^i(A'') \arrow[r, "\delta^i"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & T^{i+1}(A') \arrow[d] \\
T^i(B'') \arrow[r, "\delta^i"] & T^{i+1}(B')
\end{tikzcd}$$

Def. δ-functor $\{T^i\}: \mathcal{C} \to \mathcal{D}$ is **universal** if given any other δ-functor $\{T'^i\}: \mathcal{C} \to \mathcal{D}$ and nat'l transf $f_0: T^0 \to (T')^0$, $\exists!$ sequence

$f_i: T^i \to (T')^i$ extending f_0

commuting with δ_i for each ses

Def An additive $F: \mathcal{C} \to \mathcal{D}$ is **effaceable** if $\forall A \in \mathcal{C}$ $\exists$ $A \xrightarrow{u} B$ mono

$F(u) = 0$

Coeffaceable if $\forall A$ $\exists$ $D \xrightarrow{v} A$ epi

$F(v) = 0$

Thm $T = \{T^i\}_{i \geq 0}$ cov δ-functor

If T^i is effaceable $\forall i > 0$

then T is universal

Cor Assume $\mathcal{C}$ has enough injectives

Then if $F : \mathcal{C} \to \mathcal{D}$ left exact, then

$\{R^iF\}_{i \geq 0}$ univ δ-funct. $R^0 F \simeq F$

Conversely, if $T = \{T^i\}_{i \geq 0}$ any univ

δ-functor and T^0 left exact,

then $T^i \simeq R^i(T^0)$

Pf: see text

Derived Categories

$\mathcal{C}, \quad C^\cdot(\mathcal{C})$

$K^\cdot(\mathcal{C})$ = homotopy category

$\mathrm{Ob}\, K^\cdot(\mathcal{C}) = \mathrm{Ob}(C^\cdot(\mathcal{C}))$

$\mathrm{Mor}_{K^\cdot(\mathcal{C})}(A^\cdot, B^\cdot) = \mathrm{Mor}_{C^\cdot(\mathcal{C})}(A^\cdot, B^\cdot) \big/ \simeq$ homotopic

Translation functor $\quad T : K^\cdot(\mathcal{C}) \longrightarrow K^\cdot(\mathcal{C})$

$(T(A^\cdot))^i = A^{i+1} \qquad d_{T(A^\cdot)} = -d_{A^\cdot}$

$a : A^\cdot \to B^\cdot \qquad T(a)^i = a^{i+1}$

$T(a^\cdot) : T(A^\cdot) \to T(B^\cdot)$

Alternate notation: $\quad A^\cdot[1] = T(A^\cdot)$

"shift by 1 to left"

Cones

$\alpha : A^\cdot \to B^\cdot$

$\mathrm{Cone}(\alpha) \in K^\cdot(\mathcal{C})$

As a complex $\mathrm{Cone}(\alpha) = T(A^\cdot) \oplus B^\cdot$

$$\begin{tikzcd} B^\cdot \arrow[d] \\ T(A^\cdot) \oplus B^\cdot \arrow[d] \\ T(A^\cdot) \end{tikzcd}$$

Differential

$$d_{\mathrm{Cone}(\alpha)} = \begin{pmatrix} d_{T(A^\cdot)} & 0 \\ T(\alpha) & d_{B^\cdot} \end{pmatrix}$$

$$d_{\mathrm{Cone}(\alpha)} : \underset{\underset{A^{i+1} \oplus B^{i}}{\parallel}}{\mathrm{Cone}(\alpha)^{i}} \longrightarrow \underset{\underset{A^{i+2} \oplus B^{i+1}}{\parallel}}{\mathrm{Cone}(\alpha)^{i+1}}$$

$$A^\cdot \xrightarrow{\alpha} B^\cdot \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow T(A^\cdot)$$

distinguished triangle

Example of distinguished triangle

$$\begin{tikzcd} A^\cdot \arrow[rr] & & B^\cdot \arrow[dl] \\ & \mathrm{Cone}(\alpha) \arrow[ul, "T\,+1"] & \end{tikzcd}$$

Recall $h^i : C^\cdot(\mathcal{C}) \to \mathcal{C}$

Induces $h^i : K^\cdot(\mathcal{C}) \to \mathcal{C}$

Def $f : A^\cdot \to B^\cdot$ is a quasi-isomorphism (qi) if $\forall i$

$$h^i(f) : h^i(A^\cdot) \xrightarrow{\sim} h^i(B^\cdot)$$

“Invert qi” to get derived category

Ex. $0 \to A \to B \to C \to 0$ s.e.s.

$$\begin{tikzcd} & 0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & B \arrow[r] & C \arrow[r] & 0 & \end{tikzcd}$$

$h^0(G^\cdot) = A$

$h^1(G^\cdot) = B$

$$\begin{tikzcd} G^\cdot \arrow[d, "qi"'] & A \\ F^\cdot \arrow[ur, dash, "invert"'] & \end{tikzcd}$$

$A = h^0(F^\cdot)$ $h^1(F^\cdot) = 0$

$D(\mathcal{C})$

$\mathrm{Ob}\, D(\mathcal{C}) = \mathrm{Ob}\,(K^{\cdot}(\mathcal{C}))$

$$\mathrm{Mor}_{D(\mathcal{C})}(A^{\cdot}, B^{\cdot}) = \left\{ \begin{tikzcd} Q^{\cdot} \arrow[d, "qis"'] \arrow[dr] & \\ A^{\cdot} & B^{\cdot} \end{tikzcd} \right\} / \sim$$

$$\begin{tikzcd} Q^{\cdot} \times_{B^{\cdot}} R^{\cdot} \arrow[d, "qis"'] \arrow[dr] \arrow[dd, bend right=60, "qis"'] & & \\ Q^{\cdot} \arrow[d, "qis"'] \arrow[dr] & R^{\cdot} \arrow[d, "qis"] \arrow[dr] & \\ A^{\cdot} & B^{\cdot} & C^{\cdot} \end{tikzcd}$$

$(\mathrm{Mor}_{D(\mathcal{C})}(A^{\cdot}, C^{\cdot})$

defines composition

Now suppose $F : \mathcal{C} \to \mathcal{D}$ left exact, $\mathcal{C}$ has enough injectives. Will define

$$RF : D(\mathcal{C}) \to D(\mathcal{D})$$

$A^\cdot \in D(\mathcal{C})$ $\exists$ $A^\cdot \xrightarrow{qi} I^\cdot$ $I^\cdot$ inject (exercise)

$A \in \mathcal{C}$ $A^\cdot =$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "qi"] & 0 & & \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

$$h^i(I^\cdot) = \begin{cases} 0 & i \neq 0 \\ A & i = 0 \end{cases}$$

$0 \to A \to I^0 \to I^1 \to \cdots$

inj res

$$RF(A^\cdot) \overset{def}{=} F(I^\cdot) \in D(\mathcal{D})$$

X, $\mathcal{F} \in Ab(X)$ III. 2

Def $\mathcal{F}$ is **flasque** if $\forall\, \underline{U \subset V} \subset X$ open, $\mathcal{F}(V) \twoheadrightarrow \mathcal{F}(\underline{U})$

Ex $\mathcal{F}$ constant sheaf on irred space X is flasque (see next page)

Lemma: $(X, \mathcal{O}_X)$ ringed $\mathcal{I} \in Mod(X)$ injective

$\implies \mathcal{I}$ flasque

Pf: $\mathcal{O}_U := j_!(\mathcal{O}_X|_U) \quad j : U \hookrightarrow X$ —

$$j_!(\mathcal{O}_X|_U)(W) = \begin{cases} \mathcal{O}_X(W) & W \subset U \\ 0 & W \not\subset U \end{cases}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_U^{(w)} \arrow[r] \arrow[d, "s"] & \mathcal{O}^{(ws)} \arrow[dl, "\exists t"] \\ & \mathcal{I} & \end{tikzcd}$$

$\operatorname{Hom}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$

Try to lift $\underline{s \in \mathcal{I}(U)}$ to $\mathcal{I}(U)$

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