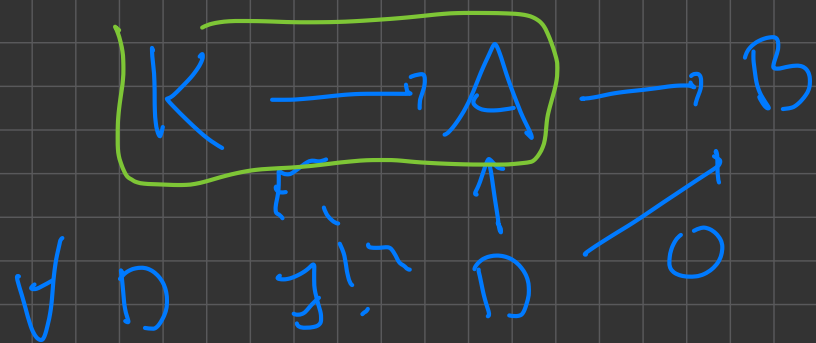


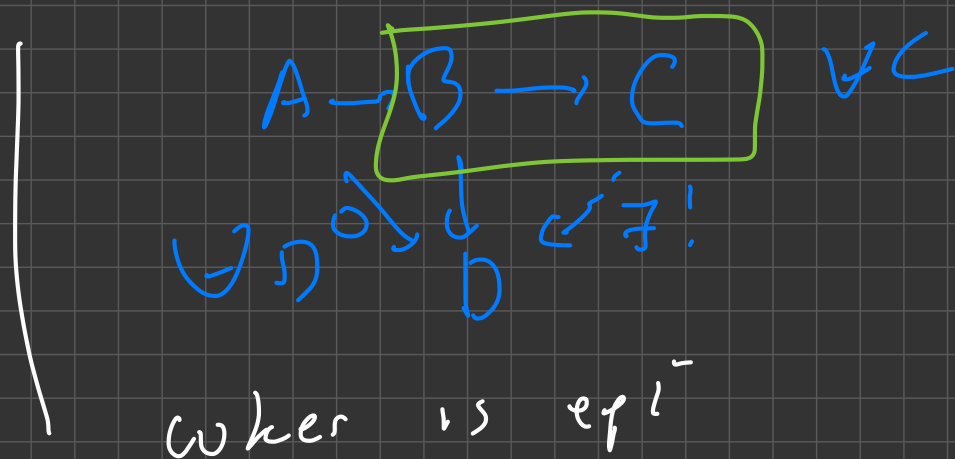
$f: A \rightarrow B$ $A, B \in \mathcal{C}$ category

kernel of f ,



kernel is a monomorphism.

cokernel of f



⊕

Idea of abelian category
Behaves like ab. grps, modules over comm ring, etc

Examples

• Ab

abelian groups

• $Mod(A)$

mod. over comm ring A

• $Ab(X)$

sheaves of ab. grps on top space X

• $Mod(X)$

(X, \mathcal{O}_X) ringed space sheaves of \mathcal{O}_X -mod.

• $QCoh(X)$

X scheme : quasi-coherent sheaf

• $Coh(X)$

X Noetherian scheme : coherent sheaves

Def An abelian category \mathcal{C} $\forall A, B \in \mathcal{C}$

• $\text{Hom}(A, B)$ ab. gp

• $\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ comp. bilin.

• \oplus exist

• Every morph. has a kernel + cokernel

• Every monomorphism is kernel of its cok.

• Every epimorphism cokernel of its ker

$$A \xrightarrow{f} B \xrightarrow{g} \text{coker } f$$

$$\ker g = \text{Im } f \subseteq A$$

if f mono-

\mathcal{C} abelian category

$C(\mathcal{C})$ of chain complexes in \mathcal{C}

$$A^\bullet \in C(\mathcal{C}) \quad \dots \rightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \rightarrow \dots$$

$d^i \circ d^{i-1} = 0$

$$h^i(A^\bullet) \in \mathcal{C}$$

$$h^i(A^\bullet) = \frac{\ker d^i}{\operatorname{im} d^{i-1}} \in \mathcal{C}$$

$$f: A^{\bullet} \rightarrow B^{\bullet} \quad \text{c.c.}(\mathcal{C})$$

$$\begin{array}{c} \rightarrow A^{\bullet} \xrightarrow{d_A^{\bullet}} A^{\bullet} \\ \downarrow f^{\bullet} \quad \downarrow f^{\bullet} \\ \rightarrow B^{\bullet} \xrightarrow{d_B^{\bullet}} B^{\bullet} \end{array}$$

$$h^{\bullet}(f): h^{\bullet}(A^{\bullet}) \rightarrow h^{\bullet}(B^{\bullet})$$

$$f^{\bullet} d_A^{\bullet} = d_B^{\bullet} f^{\bullet}$$

$$\left. \begin{array}{ccc} \ker d_A^{\bullet} & \xrightarrow{f^{\bullet}} & \ker d_B^{\bullet} \\ \text{Im } d_A^{\bullet-1} & \xrightarrow{f^{\bullet}} & \text{Im } d_B^{\bullet-1} \end{array} \right\}$$

$$\text{induce } h^{\bullet}(f) \text{ on } \frac{\ker d_A^{\bullet}}{\text{Im } d_A^{\bullet-1}}$$

Short exact sequence of complexes

$$0 \rightarrow A^{\bullet} \rightarrow B^{\bullet} \rightarrow C^{\bullet} \rightarrow 0$$

s.e.s.

$$0 \rightarrow A^{\bullet} \rightarrow B^{\bullet} \rightarrow C^{\bullet} \rightarrow 0$$

$$\delta^i: h^{\bullet}(C^{\bullet}) \rightarrow h^{\bullet+1}(A^{\bullet})$$

Prop $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ seq

gives rise to long exact sequence

$$\dots \rightarrow H^l(A) \rightarrow H^l(B) \rightarrow H^l(C) \rightarrow \dots$$

$$\dots \rightarrow H^{l+1}(A) \rightarrow H^{l+1}(B) \rightarrow H^{l+1}(C) \rightarrow \dots$$

$$\dots \rightarrow H^{l+2}(A) \rightarrow \dots$$

Pf (1) Can be proven directly from the axioms for abelian categories (hard!)

(2) Use Freyd embedding theorem and diagram chasing

$\mathcal{A} \subset \text{Mod}(A)$ for some A

Homotopy

$$f, g: A \rightarrow B$$

$$f \sim g$$



$$f^l - g^l = d^{l-1} k^l + k^{l+1} d^l$$

Prop: If $f \sim g$, then $h^l(f) = h^l(g)$

Proof: $h^l(A) \ni [h]$ is in $\ker d_A^l$

$h^l(f) - h^l(g)$ induced by $d^{l-1} k^l + k^{l+1} d^l$

Apply to h $\left(\underbrace{d^{l-1} k^l h}_{\in \ker d^{l-1}} + \underbrace{k^{l+1} d^l h}_0 \right) =$

$$[d^{l-1} k^l h] = 0 \text{ in } h^l(B)$$

$$h^l(f) = h^l(g)$$

\mathcal{C}, \mathcal{D} abelian categories $F: \mathcal{C} \rightarrow \mathcal{D}$
covariant functor

Def F is additive if $\forall A, B \in \mathcal{C}$

$$\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$$

is a homom. of ab. grps

F is left exact if it is additive and

for all seq $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ seq

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \quad \text{exact}$$

right exact

$$F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0 \text{ exact}$$

exact

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0 \text{ exact}$$

For contravariant functors

$$0 \rightarrow F(C) \rightarrow F(B) \rightarrow F(A) \text{ exact (left exact)}$$

$$F(C) \rightarrow F(B) \rightarrow F(A) \rightarrow 0 \text{ right exact}$$

Ex

$$\text{Hom}(\cdot, A) : \mathbb{Q}^0 \rightarrow \mathbb{Q} \text{ left exact}$$

$$\text{Hom}(A, \cdot) : \mathbb{Q} \rightarrow \mathbb{Q} \text{ left exact}$$

\mathcal{C} abelian category



Def $I \in \mathcal{C}$
 $\text{Hom}(-, I)$

injective \iff

exact $0 \rightarrow A \rightarrow B$
 $\text{Hom}(B, I) \rightarrow \text{Hom}(A, I) \rightarrow 0$

Def $A \in \mathcal{C}$ injective resolution of A

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots \quad I^0 \text{ injective}$$

exact

Ex: $\mathcal{C} = \text{Ab}$

Def: $G \in \text{Ab}$ is divisible \iff $\forall g \in G \forall n \in \mathbb{Z}_+ \exists h \in G \text{ s.t. } g = nh$

Thm G is injective $\iff G$ is divisible

PC Suppose G not divisible.

Pick $g \in G$, $n \in \mathbb{Z}$ s.t. $\nexists h \in G$

$$g = nh$$

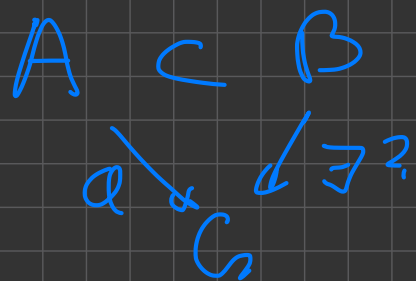
$$G \hookrightarrow \mathbb{Z} \subset \mathbb{Z} \cdot \left(\frac{1}{n}\right) \subset \mathbb{Q}$$

$$\begin{array}{ccc} & \searrow & \\ & \downarrow & \\ & g \in G & \\ & \swarrow & \\ & h & \end{array} \quad \begin{array}{c} \mathbb{Z} \cdot \left(\frac{1}{n}\right) \\ \downarrow \\ \mathbb{Z} \end{array} \quad \begin{array}{c} \mathbb{Q} \\ \downarrow \\ \mathbb{Z} \end{array} \quad \begin{array}{c} 1 = n \cdot \left(\frac{1}{n}\right) \\ \downarrow \\ 1 \end{array}$$

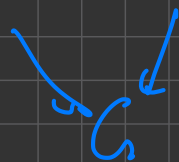
d. $g = nh$

So g not divisible.

Suppose G divisible, show G injective



By a Zorn's Lemma argument, suffices
to show $\exists H, A \subsetneq H \subset B$



Will show existence of H

Pick

$$x \in B - A$$

(1)

Suppose $nx \notin A \quad \forall n \in \mathbb{Z}_+$

Then

$$A \subsetneq H = A + \mathbb{Z}x \subset B$$



(2)

Suppose $nx \in A, n \in \mathbb{Z}_+$ minimal

Pick $g \in G, ng \equiv q(nx)$

$$A \oplus \mathbb{Z} \xrightarrow{p} G \quad p(a, m) \equiv q(a) + mng$$



$$B \supseteq \ker \alpha' \subset \ker p$$

$$\alpha'(a, m) = q(a) + mx$$

Prop $G \in \text{Ab}$. Then \exists injective H ,
 $G \subset H$

Pf (sketch)

$$\boxed{\begin{array}{l} \mathbb{Z} \subset \mathbb{Q} \\ \mathbb{Z}/n\mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \end{array}} \quad \mapsto \frac{1}{n}$$

$$g \in G \neq 0 \quad \mathbb{Z}g \subset G \quad \ni g$$

$$I_g := \mathbb{Q} \subset \mathbb{Q}/\mathbb{Z} \quad \xleftarrow{\quad} \mathbb{Z}g \neq 0$$

$$G \quad \xleftarrow{\quad} \mathbb{Z}g \quad \xrightarrow{\quad} I_g$$

$g \in G \neq 0$

1a)

\mathcal{C} abelian category

Def \mathcal{C} has enough injectives

$$\forall A \in \mathcal{C} \quad \exists \text{ (inj) } I \in \mathcal{C} \quad A \hookrightarrow I$$

Examples

$\text{Ab}, \text{Mod}(R), \text{Ab}(X), \text{Mod}(X), \text{Vect}(X), \text{Vect}(k)$

Lemma \mathcal{C} has enough injectives \implies
every $A \in \mathcal{C}$ has an inj. res.

Pf

$$\begin{array}{ccccccc} 0 \rightarrow A & \xrightarrow{\alpha} & I^0 & \rightarrow & I^1 & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ & & I^0 & \xrightarrow{\text{inj}} & I^0 / \text{Im } \alpha & \xrightarrow{\cong} & I^1 \end{array}$$

Lemma: $A \rightarrow I^{\bullet}, A \rightarrow J^{\bullet}$
 two injective resolutions. Then

I^{\bullet} and J^{\bullet} are homotopy equivalent

PC (sketch)

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \xrightarrow{f} & I^0 & \xrightarrow{d} & I^1 & \rightarrow & I^2 \\
 & & \parallel & & \downarrow d & \nearrow f & \downarrow d & & \downarrow d \\
 0 & \rightarrow & A & \xrightarrow{g} & J^0 & \rightarrow & J^1 & \rightarrow & J^2
 \end{array}$$

$$\begin{array}{ccccccc}
 0 \rightarrow A & \rightarrow & I^0 & \rightarrow & I^1 & \rightarrow & \dots \\
 & & \parallel & & \downarrow fg & & \downarrow fg \\
 0 \rightarrow A & \rightarrow & J^0 & \rightarrow & J^1 & \rightarrow & \dots
 \end{array}$$

hom to 1.

$F: \mathcal{C} \rightarrow \mathcal{D}$ covariant left exact

$\forall A \in \mathcal{C}$ pick I^\bullet injective res of A

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$$

$$R^i F(A) := h^i(F(I^\bullet))$$

$$h^i(F(I^\bullet)) = \ker d^i / \ker d^{i-1} \in \mathcal{D}$$

$$0 \rightarrow F(I^0) \xrightarrow{d^0} F(I^1) \xrightarrow{d^1} F(I^2) \rightarrow \dots$$

$$\begin{array}{ccc} 0 \rightarrow A \rightarrow I^\bullet \\ \parallel \\ 0 \rightarrow A \rightarrow J^\bullet \end{array}$$

$$\begin{array}{c} F(I^\bullet) \\ \downarrow \\ F(J^\bullet) \end{array}$$

$$\begin{array}{c} h^i(F(I^\bullet)) \\ \uparrow \downarrow \\ h^i(F(J^\bullet)) \end{array}$$

diff choices of I^\bullet
give isom $R^i F(A)$

$$R^0 F \cong F$$

Ex: X top space $F \in \text{Ab}(X)$

$$r(X, \cdot) : \text{Ab}(X) \rightarrow \text{Ab}$$

$$H^i(X, F) := R^i r(X, F) \in \text{Ab}$$

Ex (X, \mathcal{O}_X) ringed space (e.g. scheme)

$$r(X, \cdot) : \text{Mod}(X) \rightarrow \text{Ab}$$

\cup
 $\text{Ab}(X)$

Prop: derived functors agree

Then X Noether top space, $\dim X = n$, $F \in \text{Ab}(k)$

then $H^c(X, F) = 0 \quad L > n$

Ex X scheme, $\dim X = 0$ Noether

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0 \quad \text{exact}$$

$$\Rightarrow \begin{array}{c} 0 \rightarrow \Gamma(X, F') \rightarrow \Gamma(X, F) \rightarrow \Gamma(X, F'') \rightarrow 0 \\ \parallel \qquad \qquad \qquad \parallel \\ 0 \rightarrow H^0(X, F') \rightarrow H^0(X, F) \rightarrow H^0(X, F'') \rightarrow 0 \end{array}$$

Ex: X curve $\mathbb{A}^1_k \implies H^i(X, F) = 0 \quad \forall i \geq 2$
 $F \in \mathcal{A}_k(X) \implies$

Will see later that if F constant
 X irred $\implies H^i(X, F) = 0 \quad \forall i \geq 1$

Warning: If X/\mathbb{C} . Zariski top
Analytic top

Then \mathcal{C} ab cat with enough injectives, \mathcal{D} abelian

$F: \mathcal{C} \rightarrow \mathcal{D}$ covariant, left exact. Then

$R^i F: \mathcal{C} \rightarrow \mathcal{D}$ addition

$F \cong R^0 F$

\forall seq $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \in \mathcal{C}$ and $0 \rightarrow$ has

$\delta^i: R^i F(A'') \rightarrow R^{i+1} F(A')$ and l.e.s.

$0 \rightarrow R^0 F(A') \rightarrow R^0 F(A) \rightarrow R^0 F(A'') \xrightarrow{\delta^0} R^1 F(A') \rightarrow R^1 F(A) \rightarrow \dots$ *

Given morphism of seq, f_i give

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \rightarrow$$

$$0 \rightarrow A' \rightarrow B \rightarrow B'' \rightarrow 0 \rightarrow$$

$$\begin{array}{ccc} R^i F(A'') & \xrightarrow{\delta^i} & R^{i+1} F(A') \\ \downarrow f & & \downarrow f \\ R^i F(B'') & \xrightarrow{\delta^i} & R^{i+1} F(B') \end{array}$$

$I \in \mathcal{C}$ injective, $0 \rightarrow I \rightarrow 0 \rightarrow 0 \rightarrow 0$

$$R^i F(I) = 0$$

Pf (some part)

$$0 \rightarrow A \rightarrow I^0 \xrightarrow{F} I^1 \xrightarrow{\text{inf}} \dots$$

$$R^0 F(A) = \ker (F(I^0) \rightarrow F(I^1)) = \ker (F(I^0) \rightarrow F(\text{inf}))$$

$$0 \rightarrow F(A) \rightarrow F(I^0) \rightarrow F(I^1) \rightarrow F(\text{inf})$$

$$\begin{array}{ccccccc}
 0 & \rightarrow & A & \rightarrow & A & \rightarrow & A'' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & (I^1)' & \rightarrow & I^0 & \rightarrow & (I^1)'' \rightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 A' & \rightarrow & A & \rightarrow & A'' \\
 \downarrow & & \downarrow & & \downarrow \\
 (I^1)' & \rightarrow & I^0 & \rightarrow & (I^1)'' \\
 \downarrow & & \downarrow & & \downarrow \\
 I^1 & \rightarrow & (I^0/I^1) & \rightarrow & I^1
 \end{array}$$

$$0 \rightarrow F((I^1)') \rightarrow F(I^0) \rightarrow F((I^1)'') \rightarrow 0 \in D \quad R^L F(A'')$$

$$R^L F(A') \cong h^i F((I^1)') \rightarrow h^L(F(I^0)) \rightarrow h^L(F((I^1)''))$$

$$\hookrightarrow h^{L+1} F((I^1)') \rightarrow R^{L+1} F(A')$$

$$0 \rightarrow I \rightarrow I \rightarrow 0$$

n) acs of I

$$R^i F(I) = h^L (F(0 \rightarrow \dots \rightarrow I \rightarrow 0 \rightarrow \dots))$$

$$\begin{array}{c}
 \cong \\
 h^L (0 \rightarrow \dots \rightarrow F(I) \rightarrow 0) \\
 \cong \\
 \begin{cases} 0 & i \neq 0 \\ F(I) & i = 0 \end{cases}
 \end{array}$$

Return to $F: \mathcal{C} \rightarrow \mathcal{D}$ left exact, \mathcal{C} enough inj.

Def: $J \in \mathcal{C}$ acyclic for F if
 $R^c F(J) = 0 \quad \forall c > 0$

Prop: Suppose $A \in \mathcal{C}$, $0 \rightarrow A \rightarrow J^0 \rightarrow J^1 \rightarrow J^2 \rightarrow \dots$
 J^c acyclic. Then

$$R^c F(A) \cong h^c(F(J^i)) \in \mathcal{D}$$

Pf (sketch) $0 \rightarrow A \rightarrow J^0 \rightarrow K^1 \rightarrow 0$ exact

$$0 \rightarrow K^1 \rightarrow J^1 \rightarrow K^2 \rightarrow 0$$

$$0 \rightarrow K^2 \rightarrow J^2 \rightarrow K^3 \rightarrow 0 \text{ exact}$$

$$\underline{R^i F(A)} \cong \underline{R^{i-1} F(K^1)} \cong R^{i-2} F(K^2) \cong \dots \cong R^i F(K^{i-1})$$

$$0 \rightarrow F(K^{i-1}) \rightarrow F(J^{i-1}) \rightarrow F(K^i) \rightarrow R^i F(K^{i-1}) \rightarrow 0$$

$$R^i F(K^{i-1}) \cong \text{coker}(F(J^{i-1}) \rightarrow F(K^i)) \cong R^i F(J^{i-1})$$

$$0 \rightarrow A \rightarrow J^0 \rightarrow K^1 \rightarrow 0$$

$$\begin{array}{c} \cong \\ \downarrow \\ R^{L_{i-1}} F(J^0) \rightarrow R^{L_{i-1}} F(K^1) \xrightarrow[\cong]{} R^L F(A) \rightarrow R^L F(J^0) \\ \parallel \\ 0 \end{array} \quad (C32)$$

Claim

$$\text{ker} (F(J^{i-1}) \rightarrow F(K^1)) \cong h^L F(J)$$

$$F(J^{i-1}) \twoheadrightarrow F(K^1) \twoheadrightarrow F(J^i)$$

$$0 \rightarrow K^{i-1} \rightarrow J^{i-1} \rightarrow K^i \rightarrow 0$$

$$\times 0 \rightarrow K^i \rightarrow J^i \rightarrow K^{i+1} \rightarrow 0$$

$$\left(\begin{array}{l} \text{ker} (F(J^i) \rightarrow F(K^{i+1})) \\ \cong \\ \text{ker} (F(J^i) \rightarrow F(J^{i+1})) \end{array} \right)$$

Derived functors satisfy a univ prop

\mathcal{C}, \mathcal{D} ab cats

Def A (covariant) δ -functor $\mathcal{C} \rightarrow \mathcal{D}$

is a collection $\{T^i\}_i = \mathcal{C} \rightarrow \mathcal{D}$ and \forall ses

$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in \mathcal{C} a morph

$\delta: T^i(A'') \rightarrow T^{i+1}(A')$ s.t

- LES $0 \rightarrow T^0(A') \rightarrow T^0(A) \rightarrow T^0(A'') \xrightarrow{\delta} T^1(A') \rightarrow \dots$
- \forall morphism of ses $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$
 $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta^i} & T^{i+1}(B') \end{array}$$

Def. δ -functor $\{T^i\}: \mathcal{C} \rightarrow \mathcal{D}$ is universal
 if given any other δ -functor $\{T'^i\}: \mathcal{C} \rightarrow \mathcal{D}$

and nat'l transf $f_0: T^0 \rightarrow (T')^0, \exists!$ sequence

$f_c: T^c \rightarrow (T')^c$ extending f_0

commuting with δ_c for each c

Def An additive $F: \mathcal{C} \rightarrow \mathcal{D}$ is effaceable

if $\forall A \in \mathcal{C} \exists A \xrightarrow{u} B$ mono

$$F(u) = 0$$

coeffaceable if $\forall A \exists D \xrightarrow{v} A$ epi

$$F(v) = 0$$

Thm $\mathcal{T} = \{T^c\}_{c \geq 0}$ cov δ -functor

\Downarrow if T^c is effaceable $\forall c > 0$

then \mathcal{T} is UNIVERSAL

Cor Assume \mathcal{C} has enough injectives

Then if $F: \mathcal{C} \rightarrow \mathcal{D}$ left exact, then

$\{R^c F\}_{c \geq 0}$ univ δ -funct. $R^0 F \cong F$

Conversely, if $\mathcal{T} = \{T^c\}_{c \geq 0}$ any univ

δ -functor and T^0 left exact,

then $T^c \cong R^c(T^0)$

Pf: see text

Derived Categories

$$\mathcal{C}, \mathcal{C}(\mathcal{C})$$

$K(\mathcal{C}) =$ homotopy category

$$\text{Ob } K(\mathcal{C}) = \text{Ob } (\mathcal{C}(\mathcal{C}))$$

$$\text{Mor}_{K(\mathcal{C})}(A, B) = \text{Mor}_{\mathcal{C}(\mathcal{C})}(A, B) / \simeq \text{homotopy}$$

Translation functor $T: K(\mathcal{C}) \rightarrow K(\mathcal{C})$

$$(T(A^i))^i = A^{i+1}$$

$$d_{T(A^i)} = -d_{A^i}$$

$$a: A^i \rightarrow B^i$$

$$T(a)^i = a^{i+1}$$

$$T(a): T(A^i) \rightarrow T(B^i)$$

Alternate notation: $A^i[1] = T(A^i)$

"shift by 1 to left"

Cones

$$Q: A^\bullet \rightarrow B^\bullet$$

$$\text{Cone}(Q) \in K(C)$$

As a complex

$$\text{Cone}(Q) = T(A^\bullet) \oplus B^\bullet$$

Differential

$$d_{\text{Cone}(Q)}$$

$$= \begin{pmatrix} d_{T(A^\bullet)} & 0 \\ \tau(Q) & d_{B^\bullet} \end{pmatrix}$$

$$\tau(A^\bullet)$$

$$d_{\text{Cone}(Q)}: \text{Cone}(Q)^i \rightarrow \text{Cone}(Q)^{i+1}$$

$$\cong A^{i+1} \oplus B^i$$

$$\cong A^{i+2} \oplus B^{i+1}$$

$$A^\bullet \xrightarrow{Q} B^\bullet \rightrightarrows \text{Cone}(Q) \xrightarrow{\cong} T(A^\bullet)$$

distinguished triangle

Example of distinguished triangle

$$A^\bullet \rightarrow B^\bullet$$

$$\tau \rightarrow \uparrow \downarrow \text{Cone}(Q)$$

Recall $h^i: C^i(\mathcal{C}) \rightarrow \mathbb{C}$

Induces $h^i: K^i(\mathcal{C}) \rightarrow \mathbb{C}$

Def $f: A \rightarrow B$ is a

quasi-isomorphism (q.i.) if $\forall i$

$$h^i(\mathcal{C}) : h^i(A) \xrightarrow{\cong} h^i(B)$$

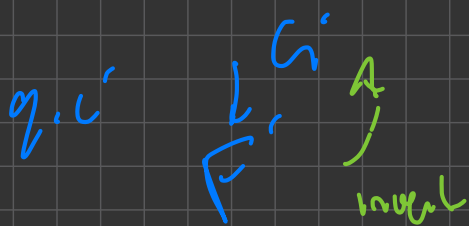
cc invent "q.i." to get derived category

Ex.

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

$$0 \rightarrow A \rightarrow 0$$

$$0 \rightarrow B \rightarrow C \rightarrow 0$$



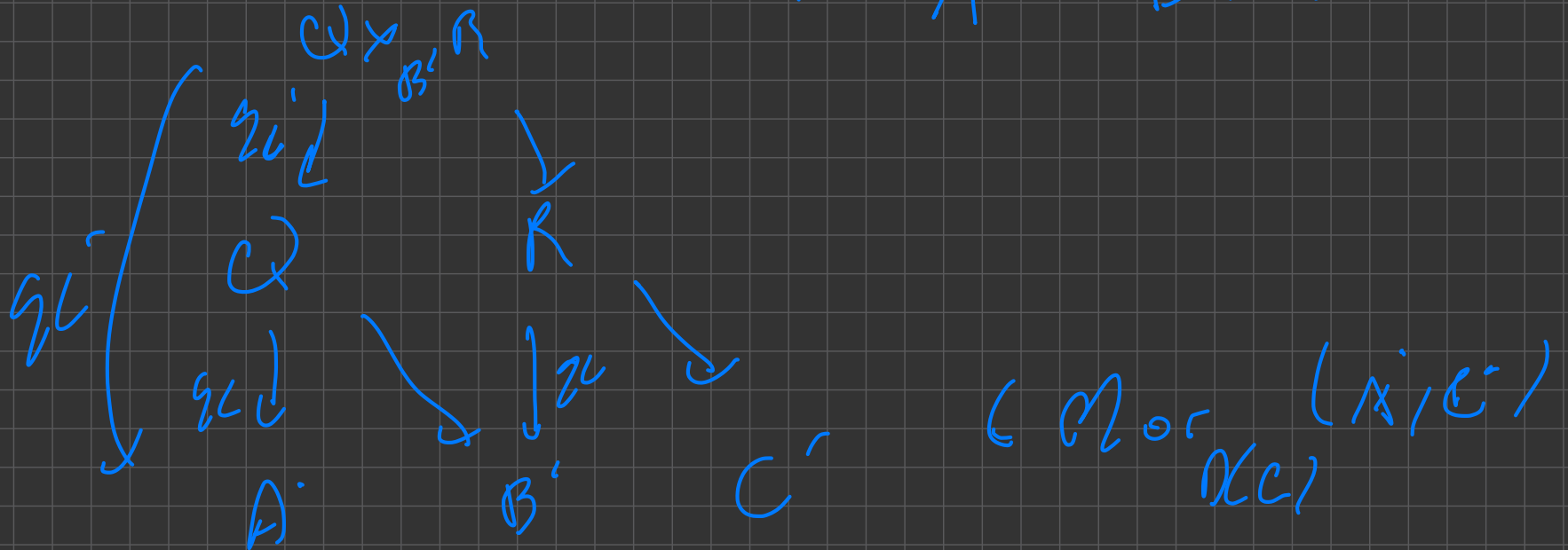
$$A = h^0(F) \quad h^i(F) = 0$$

s.e. \rightarrow
 $h^0(\mathcal{C}) = A$
 $h^i(\mathcal{C}) = B$

$D(e)$

$$0 \hookrightarrow D(e) = 0 \hookrightarrow (K(e))$$

$$\text{Mor}_{D(e)}(A', B') = \left\{ \begin{array}{c} Q \\ \text{id} \downarrow \\ A' \end{array} \rightarrow B' \right\} / \sim$$



defines composition

Now suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ left exact,

\mathcal{C} has enough injectives. Will

define

$$RF: \mathcal{D}(\mathcal{C}) \rightarrow \mathcal{D}(\mathcal{D})$$

$$A' \in \mathcal{D}(\mathcal{C}) \quad \exists A \xrightarrow{q_i} I' \quad I' \text{ injective (exact)}$$

$$A \in \mathcal{C} \quad A = 0 \rightarrow A \rightarrow 0 \quad \begin{array}{c} \downarrow q_i \\ 0 \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots \end{array}$$

$h^i(I^i) \cong A \quad \begin{array}{l} \hookrightarrow 0 \\ \hookrightarrow 0 \end{array}$

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

(1) rec

$$RF(A') \stackrel{\text{def}}{=} F(I') \in \mathcal{D}(\mathcal{D})$$

$X, F \in \text{Ab}(X)$

TIB. 2

Def F is flasque if $\forall U \subset V \subset X$

open, $F(V) \twoheadrightarrow F(U)$

Ex F constant sheaf on irred space X is flasque
(see next page)

Lemma: (X, \mathcal{O}_X) ringed $\mathcal{J} \in \text{Mod}(X)$ injective
 $\implies \mathcal{J}$ flasque

PF. $\mathcal{O}_U := j_!(\mathcal{O}_X|_U)$ $j: U \hookrightarrow X$

$$j_!(\mathcal{O}_X|_U)(W) = \begin{cases} \mathcal{O}_X(W) & W \subset U \\ 0 & W \not\subset U \end{cases}$$

$$0 \rightarrow \mathcal{O}_U^{(W)} \rightarrow \mathcal{O}_V^{(W)}$$

Try to lift $s \in \mathcal{O}_U^{(W)}$ to $\mathcal{O}_V^{(W)}$

$$\text{Hom}(\mathcal{O}_U^{(W)}, \mathcal{J}) = \mathcal{J}(W)$$

\swarrow \searrow \downarrow