Čech Cohomology

X top space

$\mathfrak{U} = \{U_\iota\}_{\iota \in I}$ open cover

Choose total ordering on I

$i_0 < i_1 < \cdots < i_p$

$U_{i_0 i_1 \cdots i_p} := U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}$

$i_0 < i_1 < \cdots < i_p$

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p})$$

Čech complex of $\mathcal{F}$

$$d : \underset{\alpha}{C^p(\mathcal{U}, \mathcal{F})} \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$$

$$(d\alpha)_{i_0 \dots i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \, \alpha_{i_0 \dots \hat{i_j} \dots i_p}\Big|_{U_{i_0 \dots i_p}}$$

Computation : $d^2 = 0$

Simplest case : $d^2 : \underset{\alpha}{C^0(\mathcal{U}, \mathcal{F})} \longrightarrow C^2(\mathcal{U}, \mathcal{F})$

$$(d^2\alpha)_{i_0 i_1 i_2} = (d\alpha)_{i_1 i_2} - (d\alpha)_{i_0 i_2} + (d\alpha)_{i_0 i_1}$$

$$= (\alpha_{i_2} - \alpha_{i_1}) - (\alpha_{i_2} - \alpha_{i_0}) + (\alpha_{i_1} - \alpha_{i_0})$$

$$= 0$$

Def: $\check{H}^p(\mathcal{U}, F) := h^p(C^\cdot(\mathcal{U}, F))$

Warning: $\check{H}^p(\mathcal{U}, F)$ is not a δ-functor

Remark. Sometimes convenient to modify def of $C^p(\mathcal{U}, F)$ to include all indices $i_0 \dots i_p$, not necessarily in increasing order

$$C^p(\mathcal{U}, F) := \left\{ \alpha = \{\alpha_{i_0 \dots i_p}\} \in \prod_{i_0 \dots i_p} F(U_{i_0 \dots i_p}) \;\middle|\; \alpha_{i_0 \dots i_j \dots i_k \dots i_p} = -\alpha_{i_0 \dots i_k \dots i_j \dots i_p} \right\}$$

$\alpha_I = 0$ if $i_j = i_k$

$\alpha_{i i'} = -\alpha_{i' i}$

clearly isomorphic to orig def.

Example $X = \mathbb{P}^1_k$, $F = \mathcal{O}(-2)$ $\mathcal{U} = \{U_0, U_1\}$

$$\check{H}^1(\mathcal{U}, \mathcal{O}(-2)) \cong k$$

$$\mathcal{O}(-2)(U_0) \oplus \mathcal{O}(-2)(U_1) \qquad \mathcal{O}(-2)(U_{01})$$

$$0 \to C^0(\mathcal{U}, \mathcal{O}(-2)) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{O}(-2)) \xrightarrow{d} 0$$

$$C^0(\mathcal{U}, \mathcal{O}(-2)) = \left\{ \left(\frac{f}{X_0^d}, \frac{g}{X_1^e} \right) \middle| \begin{array}{l} \deg f = d-2 \\ \deg g = e-2 \end{array} \right\}$$

$$C^1(\mathcal{U}, \mathcal{O}(-2)) = \left\{ \frac{h}{(X_0X_1)^f} \middle| \begin{array}{l} \deg h \\ = f-2 \end{array} \right\}$$

$\mathbb{P}^1 = \mathrm{Proj}\, k[X_0 X_1]$ $\quad h^1(\check{C}(\mathcal{U}, \mathcal{O}(-2)))$

$$= C^1(\mathcal{U}, \mathcal{O}(-2)) / \operatorname{Im} d = \operatorname{Span}\left[\frac{1}{X_0X_1} \right]$$

$$\frac{1}{X_0X_1} = \frac{X_0^2X_1^2}{(X_0X_1)^3} \qquad \frac{X_0^4}{(X_0X_1)^3} = \frac{X_0}{X_1^3} \in \operatorname{Im} d$$

Lemma Have natural isomorphism $\check{H}^0(\mathcal{U}, F) \cong \Gamma(X, F)$

Pf $\check{H}^0(\mathcal{U}, F) = \ker\left(C^0(\mathcal{U}, F) \xrightarrow{d} C^1(\mathcal{U}, F)\right)$

Let $\alpha = \{\alpha_i\} \in \check{H}^0(\mathcal{U}, F)$ $\quad \alpha_i \in F(U_i)$

$$\Rightarrow (d\alpha)_{ij} = \alpha_j|_{U_{ij}} - \alpha_i|_{U_{ij}} = 0$$

in the sequel, typically abbreviate as $\alpha_j - \alpha_i$

$\therefore$ $\alpha_i = \alpha_j$ on U_{ij}

so glue to $\tilde{\alpha} \in \Gamma(X, F)$

Conversely, given $\tilde{\alpha} \in \Gamma(X, F)$

define $\alpha \in \check{H}^0(\mathcal{U}, F)$ by

$\alpha_i = \tilde{\alpha}|_{U_i}$ $\quad F(U_i)$ $\quad d\alpha = 0$

Čech cohomology can be sheafified:

$$f_{i_0 \dots i_p} : U_{i_0 \dots i_p} \hookrightarrow X$$

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} (f_{i_0 \dots i_p})_* \mathcal{F}|_{U_{i_0 \dots i_p}} \in Ab(X)$$

a sheaf on X

Have morphism of sheaves

$$d : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$$

defined by same formula as before

$$\begin{array}{ccccc} U_{i_0 \dots \hat{i_j} \dots i_p} & \overset{f}{\hookrightarrow} & X & & f_*(U_{i_0 \dots \hat{i_j}}) \\ \cup & & & & \downarrow \text{res} \\ U_{i_0 \dots i_p} & \overset{f}{\hookrightarrow} & & & f_*(U_{i_0 \dots i_p}) \end{array}$$

Computation: $\Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F})$

Lemma $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$ (Čech resolution)

Lemma Have a natural map, functorial in $\mathcal{F}$

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

Proofs deferred

Main Thm X Noetherian, separated scheme

$\mathcal{U}$ affine

$F \in \mathrm{Qco}(X)$

Then $\check{H}^p(\mathcal{U}, F) \longrightarrow H^p(X, F)$

is an isom.

Pf: See text

Cor $H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong k$

Application

Recall

$\mathrm{Pic}(X)$ group of isomorphism classes of invertible sheaves on X

Ex 4.5 $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\times)$

$\mathrm{Pic}(\mathbb{P}^1_k) \simeq \mathbb{Z}$ $\{\mathcal{O}(n)\}$

What about $\mathrm{Pic}(X)$ with $X^{red} \simeq \mathbb{P}^1$?

S smooth projective surface over k

$C \subset S$ smooth projective curve

$$\mathcal{I} = \mathcal{I}_{C/S} \simeq \mathcal{O}_S(-C)$$

$$\tilde{C} = (C, \mathcal{O}_S/\mathcal{I}^2)$$ "1st order neighborhood of C in S" $\tilde{C}_{red} = C$

$$\mathcal{I}_{C,\tilde{C}} = \tilde{\mathcal{I}} \subset \mathcal{O}_{\tilde{C}} \qquad \tilde{\mathcal{I}}^2 = 0$$

$\tilde{\mathcal{I}}$ can be identified with the $\mathcal{O}_C$ - module $\mathcal{I}/\mathcal{I}^2$, an invertible sheaf

$$0 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{f \mapsto df} \Omega^1_S|_C \to \Omega^1_C \to 0 \qquad \Omega^1_C \simeq \omega_C$$

$$\omega_S|_C \simeq \Lambda^2 \Omega^1_S|_C \simeq \mathcal{I}/\mathcal{I}^2 \otimes \omega_C$$

$$\boxed{\mathcal{I}/\mathcal{I}^2 \simeq \omega_S|_C \otimes \omega_C^*}$$

$$0 \longrightarrow \tilde{\mathcal{D}} \longrightarrow \mathcal{O}^*_{\tilde{C}} \xrightarrow{\text{rest}} \mathcal{O}^*_C \longrightarrow 0 \qquad \operatorname{Pic} \tilde{C} = H^1(\tilde{C}, \mathcal{O}^*_{\tilde{C}})$$

$$f \longmapsto 1+f \qquad (1+f)^{-1} = 1-f \qquad f^2 = 0$$

$$f_1 + f_2 \longmapsto 1 + f_1 + f_2$$

$$= (1+f_1)(1+f_2) \quad \text{homom}$$

$$0 \longrightarrow H^0(\tilde{\mathcal{D}}) \longrightarrow \underset{k^* \hookrightarrow}{H^0(\mathcal{O}^*_{\tilde{C}})} \twoheadrightarrow H^0(\mathcal{O}_C^*) \cong k^*$$

$$\longrightarrow H^1(\tilde{\mathcal{D}}) \longrightarrow \underline{H}^1(\mathcal{O}^*_{\tilde{C}}) \longrightarrow H^1(\mathcal{O}^*_C)$$

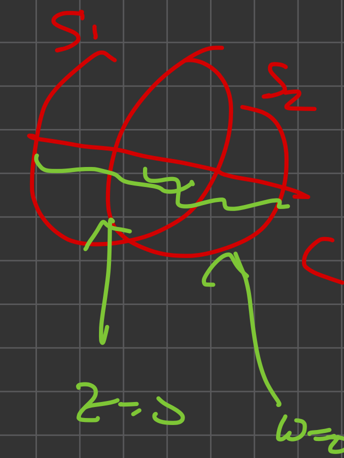
$$\longrightarrow H^2(\tilde{\mathcal{D}}) \longrightarrow$$

$$0 \neq H^1(\tilde{\mathcal{D}}) \hookrightarrow \operatorname{Pic}\underline{(\tilde{C})} \longrightarrow \operatorname{Pic}\underline{(C)}$$

Explicit example: S contains two affines

$S_0 = \operatorname{Spec} k[x,z]$ $\quad S_1 = \operatorname{Spec} k[y,w]$

$S_1 \cup S_2 \subset S$



$S_0 \supset D(x) \cong D(y) \subset S_1$

$y = x^{-1}$ $\quad w = x^{-2} z$

$C \subset S$ curve defined by

$C_0 = C \cap S_0 = V(z)$

$C_1 = C \cap S_1 = V(y)$

Identifying C with $\mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1]$

identifies C_0 with U_0

C_1 with U_1

In S_0, $I = (z)$

In S_1, $I = (w)$, so

$$\tilde{C}_0 = \tilde{C} \cap S_0 = \operatorname{Spec} k[x, z]/z^2$$

$$\tilde{C}_1 = \tilde{C} \cap S_1 = \operatorname{Spec} k[y, w]/w^2$$

$\tilde{\theta}$ inv. sheaf on C

gen by z on U_0

w on U_1

$$wx_1^2 \in \Gamma(U_1, \tilde{\mathcal{O}} \otimes \mathcal{O}(2)) = \Gamma(U_1, \tilde{\mathcal{O}}(2))$$

$$= (x^{-2} z)\, x_1^2 = \frac{x_0^2}{x_1^2} z\, x_1^2 = z x_0^2 \in \Gamma(U_0, \tilde{\mathcal{O}}(2))$$

Glues to global section $s \in \Gamma(\mathbb{P}^1, \tilde{\mathcal{O}}(2))$

In $U_1 = \mathcal{O}_+(x_1)$, x_1 is a unit

So wx_1^2 is a nowhere vanishing section on U_1 of invertible sheaf $\tilde{\mathcal{O}}(2)$

Similarly zx_0^2 nowhere vanishing section on U_0 of $\tilde{\mathcal{O}}(2)$

$\therefore$ s nowhere vanishing global section of $\tilde{\mathcal{O}}(2)$

$$\mathcal{O}_{\mathbb{P}^1} \xrightarrow[\cong]{s} \tilde{\mathcal{O}}(2)$$

$$\mathcal{O}_{\mathbb{P}^1}(-2) \xrightarrow[\cong]{\cdot s} \tilde{\mathcal{O}}$$

$$\frac{1}{x_0 x_1} \quad \check{H}^1(\mathbb{P}^1, \mathcal{O}(-2))$$

$$\Big\downarrow \qquad s \Big\downarrow z$$

$$\frac{s}{x_0 x_1} \in \check{H}^1(\mathcal{U}, \tilde{\mathcal{O}}) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{O}^\times_{\tilde{C}}) \qquad \mathcal{U} = \{U_0, U_1\}$$

$$\cong$$

$$H^1(\tilde{C}, \mathcal{O}^\times_{\tilde{C}}) \cong \mathrm{Pic}(\tilde{C})$$

$$= \frac{z x_0^2}{x_0 x_1} = z \frac{x_0}{x_1} = z x^{-1}$$

$$\check{H}^1(\mathcal{U}, \tilde{\mathcal{O}}) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{O}^\times_{\tilde{C}}) = \mathrm{Pic}(\tilde{C})$$

$$z x^{-1} \longmapsto 1 + z x^{-1}$$

Conclusion: Glue $\mathcal{O}_{\tilde{C}_0}$ on $\tilde{C}_0$ to $\mathcal{O}_{\tilde{C}_1}$ on $\tilde{C}_1$ to get nontrivial $\mathcal{L} \in \mathrm{Pic}(\tilde{C})$

Since $(1 + z x^{-1})\big|_C \overset{(z \to 0)}{=} 1$, $\mathcal{L}\big|_C \cong \mathcal{O}_C$ is trivial

$$\Gamma(X, C^{\bullet}(\mathfrak{U}, F)) = \bigoplus \Gamma(X, f_{i_0 \dots i_p *}(F|_{U_{i_0 \dots i_p}}))$$

$$= \bigoplus \Gamma(U_{i_0 \dots i_p}, F|_{U_{i_0 \dots i_p}})$$

$$= \bigoplus F(U_{i_0 \dots i_p})$$

$$= C^{\bullet}(\mathfrak{U}, F)$$

$$\mathcal{C}^0(\mathcal{U}, \mathcal{F}) = \prod f_{i*}(\mathcal{F}|_{U_i})$$

$$\mathcal{F} \longrightarrow \underbrace{f_{i*} f_i^* \mathcal{F}}_{} = f_{i*}(\mathcal{F}|_{U_i})$$

defined by adjunction

$$\mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F})$$

Lemma $0 \longrightarrow \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to$

is a resolution of $\mathcal{F}$

Pf $\mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ injective ✓

Let $x \in X$, show injective at x.

Pick $x \in U_i$

$$(\mathcal{F})_x \xrightarrow{\cong} \left((f_i)_* (\mathcal{F}|_{U_i}) \right)_x$$

$$\cap$$

$$\left((f_i)_* (\mathcal{F}|_{U_i}) \right)_x$$

$$\oplus$$

Exactness at $\mathcal{C}^p(\mathcal{U}, \mathcal{F})_x$ $\qquad p \geq 1$

Will construct homotopy

$$k : \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \longrightarrow \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$$

$$kd + dk = 1 - 0$$

$\therefore$ 1 and 0 induce same map

on $h^p(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})_x)$

so $= 0$

Will use antisymmetrization convention

$$k : C^p(\mathcal{U}, F)_x \longrightarrow C^{p-1}(\mathcal{U}, F)_x$$
$$\begin{array}{c} \Downarrow \\ \alpha \end{array}$$

Rep α by $\alpha \in \Gamma(V, C^p(\mathcal{U}, F))$

$x \in V \subset U_j$

$$(k\alpha)_{i_0 \dots i_{p-1}} := (\alpha_{j i_0 \dots i_{p-1}})_x$$

$$\boxed{((dk + kd)\alpha)}_{i_0 \dots i_p} = (dk\alpha)_{i_0 \dots i_p} + (kd\alpha)_{i_0 \dots i_p}$$

$$= \sum_{r=0}^{p} (-1)^r (k\alpha)_{i_0 \dots \hat{i_r} \dots i_p} + (d\alpha)_{j i_0 \dots i_p}$$

$$= \underbrace{\sum_{r=0}^{p} (-1)^r \alpha_{j i_0 \dots \hat{i_r} \dots i_p} + \sum_r (-1)^{r+1} \alpha_{j i_0 \dots \hat{i_r} \dots i_p}}_{=0} + \alpha_{i_0 \dots i_p}$$

$dk + kd = \alpha$

Prop $X, \mathcal{U}$ as before, $\mathcal{F} \in Ab(X)$ flasque.

Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0 \quad \forall p > 0$

Pf $0 \longrightarrow \mathcal{F} \longrightarrow C^{\cdot}(\mathcal{U}, \mathcal{F})$

Čech resolution

$f_{i_0 \ldots i_p *} \left(\mathcal{F}|_{U_{i_0 \ldots i_p}} \right)$ flasque

flasque

$\Longrightarrow$

$\Longrightarrow$ can use Čech resolution to compute $H^p(X, \mathcal{F})$

$= H^p(X, \mathcal{F}) = h^p(\Gamma(X, \quad))$

$=$

Lemma: $X, \mathcal{U}$ as before, $p \geq 0$

Have natural map $\check{H}^p(\mathcal{U}, F) \to H^p(X, F)$

functorial in F

Pf

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] & I^\bullet & \text{inj res} \\
0 \arrow[r] & F \arrow[r] \arrow[u, equal] & C^\bullet(\mathcal{U}, F) \arrow[u, "\exists"'] & \text{Čech res}
\end{tikzcd}$$

$$\begin{tikzcd}
h^p(C^\bullet(\mathcal{U}, F)) \arrow[r] & h^p(I^\bullet) \\
\check{H}^p(\mathcal{U}, F) \arrow[r] \arrow[u, equal] & H^p(X, F) \arrow[u, "\cong"']
\end{tikzcd}$$