Schemes ([H] Ch II.2 plus material from [V] Ch 3-4)

Conventions about rings in this course

- All rings are commutative with 1
- All homomorphisms of rings take 1 to 1

Classical algebraic geometry example

$X = \mathbb{A}^2_k$ $\qquad$ $A(X)$ = ring of regular fcns on X

affine variety $\qquad$ $= k[x,y]$

Given $A(X)$, recover X as set of max ideals

$$\mathbb{A}^2_k = X \longleftrightarrow \{\text{max ideals of } k[x,y]\}$$

$$\qquad\quad \psi \qquad\qquad\qquad\qquad\quad \|$$

$$(a,b) \longleftrightarrow \{(x-a,\, y-b)\}$$

Affine scheme A ring

$\mathrm{Spec}(A)$ is a top. space with a sheaf of rings

As a set $\mathrm{Spec}(A) = \{\mathfrak{p} \subset A \text{ prime ideal}\}$

Let $\mathfrak{a} \subset A$ ideal

$$V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{p} \supset \mathfrak{a}\} \subset \mathrm{Spec}(A)$$

Lemma (a) If $\mathfrak{a}, \mathfrak{b} \subset A$ ideals, then

$$V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$$

(b) If $\{\mathfrak{a}_i\}$ any set of ideals of A, then

$$V\left(\sum \mathfrak{a}_i\right) = \bigcap_i V(\mathfrak{a}_i)$$

(c) $V(\mathfrak{a}) \subset V(\mathfrak{b}) \Longleftrightarrow \sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}}$

[Recall $\sqrt{\mathfrak{a}} = \{f \in A \mid f^n \in \mathfrak{a}, \overset{\text{some}}{n \geq 1}\} = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$, $\mathfrak{p}$ prime]

Partial Proof

Claim

(a) $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b}) \iff \mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b} \iff \mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$

$$\Updownarrow$$

$$\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$$

$\Leftarrow$ ✓ Pf of claim: $\mathfrak{p} \supseteq \mathfrak{a} \supseteq \mathfrak{ab} \Rightarrow \mathfrak{p} \supseteq \mathfrak{ab}$

$\Rightarrow$ Suppose $\mathfrak{p} \supseteq \mathfrak{ab}$ but $\mathfrak{p} \not\supseteq \mathfrak{a}$, $\mathfrak{p} \not\supseteq \mathfrak{b}$.

$a \in \mathfrak{a} - \mathfrak{p}$ $\quad b \in \mathfrak{b} - \mathfrak{p}$

$ab \in \mathfrak{ab} - \mathfrak{p} = \emptyset$

(b) See text

(c) $V(\mathfrak{a}) \subset V(\mathfrak{b})$. Then $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a}),\, \mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} \supseteq \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q} = \sqrt{\mathfrak{b}}$

Conversely, if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$, let $\mathfrak{p} \in V(\mathfrak{a})$

$\Rightarrow \mathfrak{a} \subseteq \mathfrak{p} \Rightarrow \sqrt{\mathfrak{a}} \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \in V(\mathfrak{b})$

$$\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}} \supseteq \mathfrak{b}$$

Cor: $\{V(\mathfrak{a})\}$ are the closed sets of a topology on $\operatorname{Spec}(A)$, the Zariski Topology

Pf: Follows from a), b), $\operatorname{Spec}(A) = V(0)$, $\emptyset = V((1))$

To complete the def. of $\operatorname{Spec} A$, need a sheaf of rings $\mathcal{O} = \mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$

Need ring $\mathcal{O}(U)$ for each $U \subset \operatorname{Spec} A$ open and restriction maps

Recall $A_{\mathfrak{p}} = S^{-1}A$, $S = A - \mathfrak{p}$

$$= \left\{ \frac{a}{b} \;\middle|\; \begin{matrix} a \in A \\ b \notin \mathfrak{p} \end{matrix} \right\} \Big/ \sim$$

$$\mathcal{O}(U) = \left\{ s : U \longrightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \;\middle|\; s(\mathfrak{p}) \in A_{\mathfrak{p}},\ \forall \mathfrak{p} \in U \right.$$

and $\forall \mathfrak{p}$ $\exists V$, $\mathfrak{p} \in V \subset U$, $a, f \in A$, $f \notin \mathfrak{q}$ $\forall \mathfrak{q} \in V$ s.t.

$$\left. s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}} \right\}$$

$\mathcal{O}(U)$ is a ring

If $U' \subset U$, define $\rho_{UU'} : \mathcal{O}(U) \to \mathcal{O}(U')$

$\underline{\mathcal{O} \text{ is a sheaf}}$

$(\mathrm{Spec}\,A, \mathcal{O})$ spectrum of A

$f \in A$ Important definition: $D(f) = \operatorname{Spec} A - V((f))$ open. The $\{D(f) \mid f \in A\}$ basis for topology of $\operatorname{Spec}(A)$ (see text & [V 3.5])

Recall from algebra: $A_f = S^{-1}A$, $S = \{1, f, f^2, \dots\}$

$= \left\{ \frac{a}{f^n} \right\} / \sim$

Prop A ring $(\operatorname{Spec} A, \mathcal{O})$ its spectrum

(a) $\forall \mathfrak{p} \in \operatorname{Spec} A$, $\mathcal{O}_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$

(b) $\forall f \in A$, $\mathcal{O}(D(f)) \simeq A_f$

(c) $\Gamma(\operatorname{Spec} A, \mathcal{O}) := \mathcal{O}(\operatorname{Spec} A) \simeq A$

$$\Gamma(\operatorname{Spec} A, \mathcal{O}_X)$$

Pf (sketch) $\quad \mathfrak{q} \in D(f) \Leftrightarrow f \notin \mathfrak{q}$

(a) See text $\quad \operatorname{Spec} A$

(b) $\Rightarrow$ (c) Put $f = 1$, so that $D(1) = X$, $A_1 \cong \Gamma(\mathcal{O})$

$\quad A_1 = A$

(b) Define $\psi : A_f \longrightarrow \mathcal{O}(D(f))$

$$\psi\left(\frac{a}{f^n}\right) = \left[\mathfrak{g} \in D(f) \longmapsto \frac{a}{f^n} \in A_{\mathfrak{g}}\right]$$

Check well-defined

Show ψ injective + surjective

Injective: If $\psi\left(\frac{a}{f^n}\right) = \psi\left(\frac{b}{f^m}\right)$

for any $\mathfrak{g} \in D(f)$, $\exists h \notin \mathfrak{g}$ s.t. $h(f^m a - f^n b) = 0 \in A_{\mathfrak{g}}$

$\mathfrak{a} := \operatorname{Ann}(f^m a - f^n b)$. $h \in \mathfrak{a}, h \notin \mathfrak{g} \Rightarrow \mathfrak{a} \notin \mathfrak{g}$

$\therefore V(\mathfrak{a}) \cap D(f) = \phi \quad \Rightarrow f \in \sqrt{\mathfrak{a}}$.

$f \in \sqrt{\Omega} \Rightarrow f^k \in \Omega$ some $k \geq 1$

$\Rightarrow f^k(f^m a - f^n b) = 0$

$\Rightarrow \frac{a}{f^n} = \frac{b}{f^m}$ in A_f

surjectivity harder, see text.

See also exercises in [V 3.5]

Def: Ringed space $(X, \mathcal{O}_X)$ top space + sheaf of rings

Morphism $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is pair

$(f, f^\#)$: $f : X \longrightarrow Y$ cont'l

$V \subset Y$ $f^\#(V) : \mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V)$ $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ morphism

$f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$

$(X, \mathcal{O}_X)$ locally ringed space : $\forall p \in X$, $\mathcal{O}_{X,p}$ is a local ring

Given $p \in X$ get induced $f^\#_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$

$$\mathcal{O}_{Y,f(p)} = \varinjlim_{V \ni f(p)} \mathcal{O}_Y(V) \qquad \mathcal{O}_Y(V) \xrightarrow{f^\#(V)} (f_*\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V)) \longrightarrow \mathcal{O}_{X,p}$$

$$\downarrow$$

$$\mathcal{O}_{X,p}$$

$p \in f^{-1}(V)$

Locally ringed spaces form a category

Def: Let (A, m_A), (B, m_B) local rings

Homomorphism $\varphi: A \to B$ local homomorphism if

$$\varphi^{-1}(m_B) = m_A .$$

Def: $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ loc. ringed space

$$(f, f^\#): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

morphism of locally ringed spaces

- morphism of ringed spaces
- $\forall p \in X$, $f_p^\#: \mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X,p}$

local homom. of local rings

Prop (a) A ring $\Rightarrow$ $(\operatorname{Spec} A, \mathcal{O})$ loc. ringed space

(b) $\varphi : A \to B$ ring homom. determines a natural homom. of loc. ringed spaces

$$(f, f^\#) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

(c) A, B rings, then any morphism of loc. ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ induced by homomorphism $\varphi : A \to B$ as in (b).

Pf (sketch) (a) $\mathcal{O}_{\mathfrak{g}} \cong A_{\mathfrak{g}}$

(b) $f : \operatorname{Spec} B \to \operatorname{Spec}(A)$ $\quad f(\mathfrak{g}) = \varphi^{-1}(\mathfrak{g})$
$\mathfrak{g} \in$

$$f^{-1}(V(a)) = V(\varphi(a)) \qquad \text{continuous}$$

$$\forall\, \mathfrak{p} \in \operatorname{Spec} B : \quad \varphi_{\mathfrak{g}} : A_{\varphi^{-1}(\mathfrak{g})} \to B_{\mathfrak{g}}$$

$$f^{\#}(V) : \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(U))$$

$$\{ s : V \to \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}} \mid \dots \} \to \{ s : f^{-1}(U) \to \coprod_{\mathfrak{g} \in f^{-1}(U)} B_{\mathfrak{g}} \}$$

induced by $\{\varphi_{\mathfrak{g}}\}$

(c) Take global sections

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} A, \mathcal{O}) & \to & \Gamma(\operatorname{Spec} B, \mathcal{O}) \\ \wr\!\| & & \wr\!\| \\ A & & B \end{array}$$

Check this induces given morphism of loc. ringed spaces.
See text

Ex: $A = k[x, y, z]$, $B = k[t]$

$\varphi : A \to B$ $\quad x \mapsto t,\ y \mapsto t^2,\ z \mapsto t^3$

$(f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$.

Let $a \in k$, $\mathfrak{p}_a := (t-a) \in \operatorname{Spec} B$

$f((t-a)) = \varphi^{-1}((t-a)) = (x-a, y-a^2, z-a^3)$
$\in \operatorname{Spec} A$

Def
- Affine scheme: locally ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O})$ some A
- Scheme: locally ringed space $(X, \mathcal{O}_X)$ s.t. $\forall p \in X\ \exists p \in U \subset X$ s.t. $(U, \mathcal{O}_X|_U)$ is affine scheme

X underlying topological space (or $\operatorname{Sp} X$)
$\mathcal{O}_X$ structure sheaf

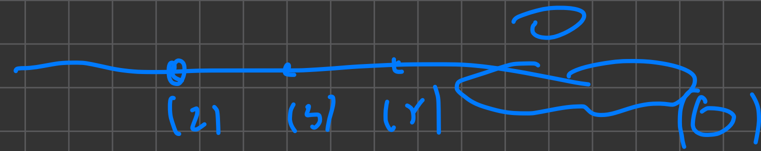
Morphism of schemes $\qquad$ category $\mathbf{Sch}$

Ex: $\operatorname{Spec} k = \{0\}$ $\mathcal{O}(\operatorname{Spec} k) = k$

$\mathbb{A}^1_k = \operatorname{Spec} k[t]$

$\mathbb{A}^1_k \supseteq (0) \subset (t-a)$

$\operatorname{Spec} \mathbb{Z}$



$X := \operatorname{Spec} k[[t]]$ (a DVR) $t_0 = (t)$ closed $t_1 = (0)$ $\overline{\{t_1\}} = X$

$\operatorname{Spec} k[x,y]$

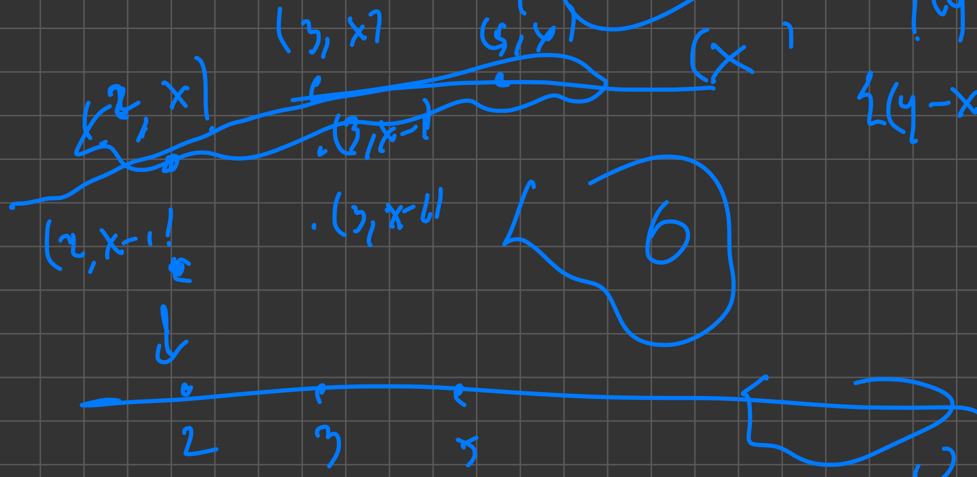
$\overline{\{(y-x^2)\}} = \{(y-x^2), (x-a, y-a^2)\}$

$\operatorname{Spec} \mathbb{Z}[x]$

$\downarrow$

$\operatorname{Spec} \mathbb{Z}$

$\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$



X_1, X_2 schemes $U_i \subset X_i$ open $i = 1, 2$

$$\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \longrightarrow (U_2, \mathcal{O}_{X_2}|_{U_2})$$

isom. of loc. ringed spaces

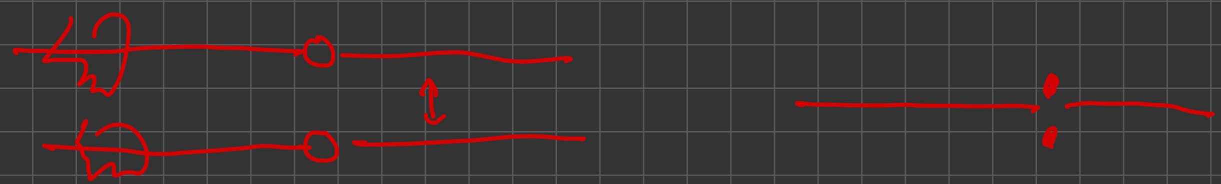
Glue $X_1 + X_2$ along U_1, U_2 using isom. φ

$$X = X_1 \amalg X_2 \,/\, p \in U_1 \sim \varphi(p) \in U_2 \qquad X_1 \hookrightarrow X, \quad X_2 \hookrightarrow X$$

see text for details

Example $\quad X_1 = X_2 = \mathbb{A}^1_k \qquad U_1 = U_2 = \mathbb{A}^1_k - \{(x)\}$

$\varphi : U_1 \to U_2$ identity



origin "doubled"

Proj

$S = \bigoplus_{n \geq 0} S_n$ graded ring. Elts of any S_n homogeneous

$S_i \cdot S_j \subset S_{i+j}$

$S_+ := \bigoplus_{n > 0} S_n$ irrelevant ideal

Def/Lemma An ideal $\mathfrak{a} \subset S$ is homogeneous if equivalently

(1) $\mathfrak{a}$ can be generated by homog. elts, or

(2) $\mathfrak{a} = \bigoplus_{n \geq 0} \mathfrak{a} \cap S_n$

Def $\operatorname{Proj} S := \{ \mathfrak{p} \subset S \mid \mathfrak{p}$ homog. ideal, $\mathfrak{p} \not\supset S_+ \}$

Next: give $\operatorname{Proj} S$ the structure of a scheme

$\mathfrak{a} \subset S$ homog. ideal

$$V(\mathfrak{a}) := \{ \mathfrak{p} \in \mathrm{Proj}\, S : \mathfrak{p} \supset \mathfrak{a} \}$$

<u>Lemma</u> • If $\mathfrak{a}, \mathfrak{b} \subset S$ homog. ideals, then

$$V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$$

• If $\{\mathfrak{a}_i\}$ any family of homog. ideals, then

$$V(\sum \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$$

Pf Essentially same as for $\mathrm{Spec}\, A$

$\therefore$ $\{V(\mathfrak{a})\}$ are closed sets of a topology on $\mathrm{Proj}(S)$.

Sheaf of rings $\mathcal{O}_{\mathrm{Proj}\, S}$

Essentially same construction as for $\mathcal{O}_{\mathrm{Spec}\, A}$

Replace $A_{\mathfrak{p}}$ by $S_{(\mathfrak{p})}$ defined as follows

$T = \{$ homog elts of S not in $\mathfrak{p} \}$

$$S_{(\mathfrak{p})} \subset T^{-1} S$$

$\{$ "homog elts of degree 0 $\}$

$$= \left\{ \frac{a}{b} \;\middle|\; \begin{array}{l} a \in S_n \\ b \in S_n - \mathfrak{p} \end{array} \text{ some } n \right\} \Big/ \sim$$

$\mathcal{O}_{\mathrm{Proj}\, S}(V) = \{ s : \mathfrak{p} \in V \longmapsto \coprod_{\mathfrak{p}} S_{(\mathfrak{p})} \mid s(\mathfrak{p}) \in S_{(\mathfrak{p})} \ldots \}$

$V \subset \mathrm{Proj}(S)$

$\exists\, \mathfrak{p} \subset W \subset U$ " $s = \frac{a}{b}$ " on V

S graded ring

Prop

- $\forall \mathfrak{p} \in \operatorname{Proj} S$, $\mathcal{O}_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$
- $\forall$ homog $f \in S_+$ homogeneous
  $D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$
  open cover of $\operatorname{Proj} S$
  $$\left(D_+(f), \mathcal{O}_{\operatorname{Proj} S}\big|_{D_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$$
  Notational Warning: here (f) does not denote the ideal gen. by f (which need not be prime). Instead $S_{(f)} \subset S_f$ is the subring of homog elts of deg 0
- $\operatorname{Proj} S$ is a scheme

Pf: Entirely analogous to the proof for Spec.

Ex: $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$

more generally, for any ring A

$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$

Exercise: $U_i = \mathbb{A}^1_k - \{0\}$ $i = 1, 2$ as before,

$U_i = D(x)$, $\mathbb{A}^1_k - \{0\} = \operatorname{Spec} k[x]_x = \operatorname{Spec} k[x, \frac{1}{x}]$

Now glue via $k[x, \frac{1}{x}] \xrightarrow{\sim} k[x, \frac{1}{x}]$

$x \longmapsto \frac{1}{x}$

Show glued scheme is isom. to $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$

$D^+(x_0) \cup D^+(x_1)$

Let S be a fixed scheme

Def A scheme over S (or S-scheme) is a scheme T together with a morphism $T \to S$

morphism of S-scheme $T \to S$, $U \to S$

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & U \arrow[d] \\ & S \end{tikzcd}$$

$Sch(S)$

(or $Sch(A)$ if $S = \operatorname{Spec} A$)

X/k $\quad X \xrightarrow{(f, f^\#)} \operatorname{Spec} k$

$$f^* : k \longrightarrow \Gamma(X, \mathcal{O}_X)$$

$$\Gamma(\mathcal{O}_{\operatorname{Spec} k})$$

k-algebra

$k = \bar{k}$

Prop $\exists$ are natural fully faithful functor

$t : Var(k) \longrightarrow Sch(k)$ s.t.

- $\forall$ $V \in Var(k)$

  V homeo to closed points of $sp(t(V))$

- Sheaf of reg functions $\mathcal{O}_V$

  is obtained from $\mathcal{O}_{t(V)}$ by restricting to V

  restricting to closed points

Pf: see text

Main examples

- $X \subset \mathbb{A}^n_k$ affine variety with ideal $I \subset k[x_1, \dots, x_n]$ then

$$t(X) = \mathrm{Spec}\left(k[x_1, \dots x_n] / I \right)$$

- $X \subset \mathbb{P}^n_k$ projective variety with homogeneous ideal $I \subset k[x_0 \dots x_n]$ then

$$t(X) = \mathrm{Proj}\left(k[x_0, \dots, x_n] / I \right)$$