Construction of associated sheaf

$\mathcal{F}$ presheaf on X, $U \subset X$ open

$$\mathcal{F}^+(U) = \Big\{ s : U \longrightarrow \coprod_{p \in U} \mathcal{F}_p \;\Big|$$

- $s(p) \in \mathcal{F}_p \quad \forall\, p \in U$
- $\forall\, P \in U \ \exists\, V$ open $P \in V \subset U$, $t \in \mathcal{F}(V)$ s.t. $\forall\, Q \in V \ s(Q) = t_Q \Big\}$

Given $\mathcal{G}$ sheaf $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ define $\varphi^+ : \mathcal{F}^+ \longrightarrow \mathcal{G}$

Let $s \in \mathcal{F}^+(U)$. To define $\varphi^+(s) \in \mathcal{G}(U)$

- Cover U by $\{V_i\}$, $t_i \in \mathcal{F}(V_i)$ s.t. $\forall\, q \in V_i$, $s(q) = (t_i)_q$
- Put $u_i = \varphi(t_i) \in \mathcal{G}(V_i)$. $\forall\, p \in V_i \cap V_j$ $(u_i)_p = \varphi_p(s_p) = (u_j)_p$
  so $u_i|_{W_p} = u_j|_{W_p}$ some $p \in W_p \subset V_i \cap V_j$ s.t. $u_i|_{W_p} = u_j|_{W_p}$
- $\{W_p\}$ cover $V_i \cap V_j$, $\mathcal{G}$ sheaf $\Longrightarrow u_i|_{V_i \cap V_j} = u_j|_{V_i \cap V_j}$
- $\exists\, u \in \mathcal{G}(U)$ s.t. $u|_{V_i} = u_i$ Put $\varphi(s) = u$

Lemma $\{D(a) \mid a \in A\}$ basis for topology of $\operatorname{Spec}(A)$

$D(a) = \operatorname{Spec} A - V((a))$

$\mathfrak{p} \in D(a)$, $\mathfrak{p} \in \operatorname{Spec} A - V((a))$

$a \notin \mathfrak{p} \iff (a) \not\subset \mathfrak{p}$

Pf Given $\mathfrak{p} \in U \subset \operatorname{Spec} A$, U open, find a s.t. $\mathfrak{p} \in D(a) \subseteq U$

$\operatorname{Spec} A - U$ closed, so $\operatorname{Spec} A - U = V(\mathfrak{a})$

$$\mathfrak{p} \in U \implies \mathfrak{p} \notin V(\mathfrak{a}) \implies \mathfrak{a} \not\subset \mathfrak{p}$$

$$\implies \exists \, \underline{a \in \mathfrak{a}}, \; a \notin \mathfrak{p}$$

Then $\mathfrak{p} \in D(a) \subset U$

$a \in \mathfrak{a} \subset \mathfrak{q}$

since $D(a) \cap V(\mathfrak{a}) = \emptyset$

$\square$

Recall A ring $f \in A$, $D(f) \subset \operatorname{Spec} A$, $\mathcal{O}$ structure sheaf of $\operatorname{Spec} A$

Prop $\mathcal{O}(D(f)) \cong A_f$

Completion of Proof

Constructed $\psi : A_f \longrightarrow \mathcal{O}(D(f))$

$$\frac{a}{f^n} \longmapsto s, \quad s(\mathfrak{p}) = \frac{a}{f^n} \in A_{\mathfrak{p}}$$

$\mathfrak{p} \in D(f) \Rightarrow f \notin \mathfrak{p}$

$\Rightarrow f^n \notin \mathfrak{p}$

We proved ψ injective

To complete the proof, show ψ surjective

Let $s \in \mathcal{O}(D(f))$, find $\frac{a}{f^n}$, $\psi\left(\frac{a}{f^n}\right) = s$.

Cover $D(f) = \bigcup V_i$ $\quad s|_{V_i} = \frac{a_i}{g_i}$ $\quad a_i, g_i \in A$

$\underbrace{g_i \notin \mathfrak{p} \ \forall\, \mathfrak{p} \in V_i}_{V_i \subseteq D(g_i)}$

WLOG $V_i = D(h_i)$, some h_i (D's basis)

$D(h_i) \subset D(g_i) \Rightarrow V((g_i)) \subseteq V((h_i))$

$\Rightarrow h_i \in \sqrt{(h_i)} \subseteq \sqrt{(g_i)}$

$\Rightarrow h_i^n = c g_i$ some $n \in \mathbb{Z}_+$, $c \in A$

$\therefore \frac{a_i}{g_i} = \frac{a_i c}{h_i^n}$

Since $D(h_i) = D(h_i^n)$, replace h_i by h_i^n, a_i by $c a_i$

WLOG $D(f) = \bigcup D(h_i)$ $\quad s|_{D(h_i)} = \frac{a_i}{h_i}$

Claim: Finitely many h_i suffice for $D(f) = \bigcup D(h_i)$

Note $D(f) \subset \bigcup D(h_i) \iff$

$V(f) \supseteq \bigcap V((h_i)) = V(\Sigma(h_i))$

$\iff f \in \sqrt{\Sigma(h_i)}$

$\iff f^n = \sum_{\text{finite } i} b_i h_i$

Fix finitely many h_i, $D(f) \subseteq D(h_1) \cup \dots \cup D(h_r)$

Next $D(h_i) \cap D(h_j) = D(h_i h_j)$ $\quad h_i h_j \notin \mathfrak{p} \iff h_i \notin \mathfrak{p}, h_j \notin \mathfrak{p}$

$\frac{a_i}{h_i}, \frac{a_j}{h_j}$ both represent s on $D(h_i h_j)$

By inj of ψ applied to $D(h_i h_j)$,

$\frac{a_i}{h_i} = \frac{a_j}{h_j} \in A_{h_i h_j} \implies (h_i h_j)^n (a_i h_j - a_j h_i) = 0$

Pick n large enough for all i, j

$$-(h_i h_j)^n (a_i h_j - a_j h_i) = 0$$

$$h_i \to h_i^{n+1} ; \quad h_j \to h_j^{n+1} ; \quad a_i \to h_i^n a_i$$

$$\frac{a_i}{h_i} \to \frac{h_i^n a_i}{h_i^{n+1}} = \frac{a_i}{h_i}$$

Then $s|_{D(h_i)} = \dfrac{a_i}{h_i}$, $\quad h_i a_j = h_j a_i$.

Recalling $f^n = \sum b_i h_i$

$$a := \sum b_i a_i$$

$$\forall j : \quad h_j a = \sum_i h_j b_i a_i = \sum_i b_i h_i a_j = f^n a_j$$

$$\frac{a}{f^n} = \frac{a_j}{h_j} \quad \text{on} \quad D(h_j)$$

$$\Rightarrow \quad \psi\left(\frac{a}{f^n}\right) = s$$

Recall A, B rings

Prop $\{A \longrightarrow B\} \longleftrightarrow \{\operatorname{Spec} B \longrightarrow \operatorname{Spec} A\}$

Completion of proof

We proved that $A \xrightarrow{\varphi} B$ determines

$$(f, f^{\#}) : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

$$f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$$

$$f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathcal{O}_{\operatorname{Spec} B}$$ induced by maps

$$\varphi_{\mathfrak{q}} : A_{\varphi^{-1}(\mathfrak{q})} \longrightarrow B_{\mathfrak{q}}$$

Conversely given $(f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$

$$\begin{array}{ccc} f^\#(\operatorname{Spec} A) : \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) & \longrightarrow & \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \\ \cong & & \cong \\ \varphi : A & \longrightarrow & B \end{array}$$

Remains to show φ induces the morphism $(f, f^\#)$ we started with

Let $\mathfrak{p} \in \operatorname{Spec} B$, have $f^\#_{\mathfrak{p}} : A_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$

Each $\varphi^\#_{\mathfrak{p}}$ compatible with $\varphi^\#(\operatorname{Spec} A)$ + localization

$$\begin{tikzcd} A \arrow[r, "\varphi = f^\#(\operatorname{Spec} A)"] \arrow[d] & B \arrow[d] \\ A_{f(\mathfrak{p})} \arrow[r, "f^\#_{\mathfrak{p}}"] & B_{\mathfrak{p}} \end{tikzcd}$$

local homom $\Longrightarrow (f^\#_{\mathfrak{p}})^{-1}(\mathfrak{p} B_{\mathfrak{p}}) = \underline{f(\mathfrak{p}) A_{f(\mathfrak{p})}}$

$\overline{\underline{\varphi}^{-1}(\mathfrak{p})} B_{\mathfrak{p}} \Longrightarrow f = \varphi$

$f^\#$ induced by

Recall useful correspondence in commutative algebra

- Given $I \subset A$ ideal

$$\{ \text{ideals of } A \text{ containing } I \} \xleftrightarrow{1-1} \{ \text{ideals of } A/I \}$$

$I \subset J \subset A \rightsquigarrow J/I \subset A/I$

$K \subset A/I \rightsquigarrow I \subset f^{-1}(K) \subset A$ where $f: A \to A/I$

Primes correspond to primes under this correspondence

Thus $V(I)$ is in bijection with $\mathrm{Spec}(A/I)$

and is in fact a homeomorphism

- Given $\mathfrak{p} \subset A$ prime

$\{$ primes of A contained in $\mathfrak{p}$ $\}$

$\xleftrightarrow{1-1}$

$\{$ primes of $A_{\mathfrak{p}}$ $\}$

$\mathfrak{q} \subset \mathfrak{p} \subset A \rightsquigarrow \mathfrak{q} A_{\mathfrak{p}} \subset A_{\mathfrak{p}}$

$\mathfrak{s} \subset A_{\mathfrak{p}} \rightsquigarrow f^{-1}(\mathfrak{s}) \subset A \quad f : A \to A_{\mathfrak{p}}$

If $f^{-1}(\mathfrak{s}) \not\subset \mathfrak{p}$, have $a \in f^{-1}(\mathfrak{s})$, $\underline{a \notin \mathfrak{p}}$

$\therefore \; f(a) \in \mathfrak{s} \qquad f(a)$ unit of $\underline{A_{\mathfrak{p}}}$

contradiction

Recall S graded ring, $f \in S^+$ homog.

Prop $\left(D_+(f), \mathcal{O}|_{D^+(f)}\right) \cong \operatorname{Spec} S_{(f)}$

Pf $D_+(f) = \operatorname{Proj} S - V((f))$ open

$\{D_+(f)\}$ cover $\operatorname{Proj} S$:

$\mathfrak{p} \in \operatorname{Proj} S \Rightarrow \mathfrak{p} \not\supset S_+ \Rightarrow \exists\, f \in S_+$ homog
$f \notin \mathfrak{p}$
$\Rightarrow \mathfrak{p} \in D^+(f)$

Fix f

$$(\varphi, \varphi^\#) : (D_+(f), \mathcal{O}_{D_+(f)}) \longrightarrow (\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec} S_{(f)}})$$

$$\mathfrak{a} \subset S \longrightarrow S_f$$

homog

$$\begin{array}{c} S_f \\ \cup \\ S_{(f)} \end{array}$$

$$\varphi(\mathfrak{a}) := (\mathfrak{a} S_f) \cap S_{(f)}$$

$\mathfrak{p} \in D_+(f) \implies$

$\varphi(\mathfrak{p})$ prime, $\varphi(\mathfrak{p}) \in \operatorname{Spec} S_{(f)}$

$\varphi : D_+(f) \longrightarrow \operatorname{Spec} S_{(f)}$ bijection by props of localization

$$\mathfrak{a} \subset \mathfrak{p} \iff \varphi(\mathfrak{a}) \subset \varphi(\mathfrak{p})$$

$\implies \varphi$ homeomorphism

Example $\quad f = x$

$y \in k[x,y] \longrightarrow k[x,y]_x$

$$\begin{array}{c} k[x,y]_x \\ \cup \\ k[x,y]_{(x)} \end{array}$$

$$\varphi((y)) = y\, k[x,y]_x \cap k[x,y]_{(x)}$$

$$= \left(\frac{y}{x}\right) \subset k[x,y]_{(x)}$$

$$k[x,y]_{(x)} = k\left[\frac{y}{x}\right]$$

To define

$$\varphi^\# : \mathcal{O}_{\mathrm{Spec}\, S_{(f)}} \longrightarrow \varphi_* \mathcal{O}_{\mathrm{Proj}\, S}\big|_{D_+(f)}$$

Note $\forall\, \mathfrak{p} \in D_+(f) \Rightarrow f \notin \mathfrak{p}$.

$f \in S_d$

$a \in S_{nd}$

$b \in S_{md}$

$b \notin \mathfrak{p}$

$\frac{a}{f^n} \in S_{(f)}$

$\frac{a/f^n}{b/f^m} = \frac{a}{b f^{n-m}}$

$\in S_{(\mathfrak{p})}$

$$(S_{(f)})_{\varphi(\mathfrak{p})} \simeq S_{(\mathfrak{p})}$$

Used to define $\mathcal{O}_{\mathrm{Spec}\, S_{(f)}}$

Used to define $\mathcal{O}_{\mathrm{Proj}(S)}\big|_{D_+(f)}$

$\square$

$k = \bar{k}$

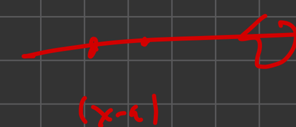
Construction of fully faithful $t: Var(k) \longrightarrow Sch(k)$

Preliminary construction: any top space X

Def X is irreducible if

$X \neq Z_1 \cup Z_2$, Z_i nonempty proper closed subset of X

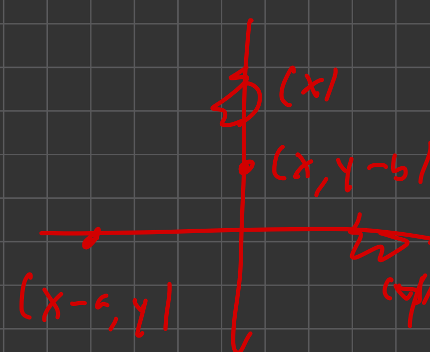
Ex $X = \operatorname{Spec} k[x]$ irred: closed sets are



$\emptyset$, $\{(x-a_1), \dots, (x-a_k)\}$, X

$X = \operatorname{Spec} k[x,y]/(xy)$ reducible

$= V(x) \cup V(y)$



$t(X) = \{$ nonempty irred closed subsets of $X\}$

$Y \subset X$ closed, then $t(Y) \subset t(X)$

$t(Y_1) \cup t(Y_2) = t(Y_1 \cup Y_2)$

irr.
$Z \subset Y_1 \cup Y_2$
$=$
$(Z \cap Y_1) \cup (Z \cap Y_2)$
$Z \subset Y_1$ or $Z \subset Y_2$

$\bigcap t(Y_i) = t(\bigcap Y_i)$

$\therefore \{t(Y) \mid Y \subset X$ closed$\}$

closed sets of a topology on $t(X)$

Given $f : X \longrightarrow Y$ continuous, define

$$t(f) : t(X) \longrightarrow t(Y) \qquad Z \subset X \longmapsto \overline{f(Z)} \subset Y$$

irred

Show $t(f)$ cont. Take $t(W) \subset t(Y)$ closed (where $W \subset Y$ closed)

$$t(f)^{-1}(t(W)) = t(f^{-1}(W)) \text{ so closed}$$

$t : \mathrm{Top} \longrightarrow \mathrm{Top}$ is a functor

$$\alpha : X \longrightarrow t(X) \qquad P \longmapsto \overline{\{P\}} \quad \text{cont.}$$

α induces bijection $\{\text{opens in } X\} \longleftrightarrow \{\text{opens in } t(X)\}$

$$X - Z \longleftrightarrow t(X) - t(Z)$$

Let $V \in \mathrm{Var}(k)$

Claim $(t(V), \alpha_* \mathcal{O}_V)$ scheme $\alpha : V \to t(V)$

Follows that $t(V) \in \mathrm{Sch}(k)$:

$$(t(V), \alpha_* \mathcal{O}_V) \longrightarrow \mathrm{Spec}\, k \iff$$

$$k \longrightarrow \alpha_*(\mathcal{O}_V)(t(V)) = \mathcal{O}_V(\alpha^{-1} t(V)) = \mathcal{O}_V(V) = A(V)$$

Pf of claim Since t bijection on opens and any variety is covered by affine varieties, may assume V affine. Will show $t(V) \cong \mathrm{Spec}\, A(V)$

Construct morphism of loc. ringed spaces

$V = \mathbb{A}^n_k$

$A(V) = k[x_1, \dots, x_n]$

$\operatorname{Spec} A(V)$

$$\beta : (V, \mathcal{O}_V) \longrightarrow \operatorname{Spec} A(V) =: X$$

$$\beta(p) = m_p \subset A(V)$$

bijection on closed pts (and homeo onto image)

For $U \subset X$ open need

$$\beta^\#(U) : \underset{\substack{\cup \\ s}}{\mathcal{O}_X(U)} \longrightarrow \mathcal{O}_V(\underset{\substack{\cup \\ p}}{\beta^{-1}(U)})$$

$\beta^\#(U)(s)$ is to be a certain k-valued fcn on $\beta^{-1}(U)$

$p \in \beta^{-1}(U)$ $\quad s \longmapsto s_p \in \mathcal{O}_{X,p} = A(V)_{m_p} \longrightarrow A(V)_{m_p} / m_p A(V)_{m_p} \cong k$

defining value of $(\beta^\#(U)(s))(p) \in k$

s locally $s = \frac{f}{g}$ $\quad f, g \in A(V)$

$g \notin p \ \forall p \in U \Longrightarrow$ nowhere 0 on $\beta^{-1}(U)$

Since s is locally of form $\frac{f}{g}$, $f, g \in A(V)$

$\Longrightarrow \beta^\#(U)(s)$ reg fcn on $\beta^{-1}(U)$

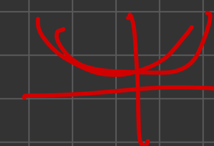
$\beta^H(U) = \mathcal{O}_X(U) \xrightarrow{\simeq} \mathcal{O}_{t(V)}(\beta^{-1}U) = \mathcal{O}_V(\alpha^{-1}\beta^{-1}U)$, $U \subset \operatorname{Spec} A(V)$

$s \in \mathcal{O}_X(U)$

$\mathcal{O}_{t(V)} = \alpha_* \mathcal{O}_V$

s locally $s|_{V_i} = \dfrac{f}{g}$, $f, g \in \operatorname{Spec} A(V)$

$U = \bigcup V_i$, $\forall \mathfrak{q} \in V_i$ $\quad g \notin \mathfrak{q}$



$\mathfrak{q} = (y - x^3)$

$p \in V$

$$t(V) \xrightarrow{\beta} \operatorname{Spec} A(V)$$

$V \supset Z$ irred closed $\longleftrightarrow \mathfrak{p}$

irred closed

$V \ni p \longleftrightarrow m_p$

$\alpha(p) = \overline{\{p\}}$

$$V \xrightarrow{\alpha} t(V) \xrightarrow{\beta} \operatorname{Spec} A(V)$$

$p \longmapsto \beta^{-1}(U) \longmapsto U$

$p \in V \qquad \beta\alpha(p) \in V_i \subset U$

$t(p) =$ image of s_p in its res. field $A(V)_{m_p}/m_p A(V) \simeq k$

Clearly $\mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{O}_V(\beta^{-1}(U))$

$\{\text{Prime of } A(U)\} \xleftrightarrow{1-1} \{\text{Irred closed subsets of } V\}$

$$\| \\ t(V)$$

$\Rightarrow \mathrm{Var}(V) \overset{U}{\cap} \longrightarrow \mathrm{Spec}\, A(U)$ $\quad \mathcal{O}_{t(V)}(U) = \mathcal{O}_V(U \cap V)$

isom of loc. ringed spaces $\quad \mathcal{O}_{\mathrm{Spec}\,A(U)}(U)$ determined by values on closed points

Finally, show $t : \mathrm{Var}(k) \longrightarrow \mathrm{Sch}(k)$ fully faithful

$V, W \in \mathrm{Var}(k)$

$$\mathrm{Hom}_{\mathrm{Var}(k)}(V, W) \xrightarrow{t} \mathrm{Hom}_{\mathrm{Sch}(k)}(t(V), t(W))$$

show bijecti

Injectivity obvious:

$t(f) : t(V) \to t(W)$ uniquely determines

$f : V \to W$ by restriction to closed pts

Given $g : t(V) \to t(W)$ must show

$g = t(f)$, some $f : V \to W$

Using uniqueness and fact that any variety can be covered by affines, WLOG assume V, W affine.

V, W affine $\Longrightarrow$

$$g : \mathcal{E}(V) \longrightarrow \mathcal{E}(W)$$
$$\quad\ \ \text{II2} \qquad\qquad \text{II2}$$
$$\operatorname{Spec} A(V) \qquad \operatorname{Spec} A(W)$$

$\therefore$ get map of rings $h := g^{\#}|_{\mathcal{E}(W)} : A(W) \longrightarrow A(V)$

$\Longrightarrow$ map of varieties

$$f : V \to W$$

Show $g = \mathcal{E}(f)$, then done.

If $P \in V$ then $g = f$ on $V \subset \mathcal{E}(V)$:

$Q = f(P)$ implies $m_P = h^{-1}(m_Q)$

same as $g(P)$

Finally $g^{\#} = (\mathcal{E}(f))^{\#}$

both describe pullbacks of regular fcns $\square$