

# Vakil, Chapter 1

## 1.2 Categories and Functors

Categories: objects and morphisms

- Collection of objects  $\text{ob}(\mathcal{C})$ ,  $\text{Ob}(\mathcal{C})$ , or just  $\mathcal{C}$
- To any objects  $A, B \in \mathcal{C}$ , a set of morphisms

$\text{Mor}(A, B)$

- Morphisms compose by a map of sets  $\forall A, B, C \in \mathcal{C}$

$$\text{Mor}(B, C) \times \text{Mor}(A, B) \xrightarrow{f} \text{Mor}(A, C)$$

Write  $f \in \text{Mor}(A, B)$  as  $f: A \rightarrow B$

Composition as  $g \circ f: A \rightarrow C$

- $\forall A \in \mathcal{C} \exists 1_A \in \text{Mor}(A, A)$  such that  
 $\forall B \in \mathcal{C}, \forall f \in \text{Mor}(A, B)$  have  $f \circ 1_A = f = 1_B \circ f$

A morphism  $f \in \text{Mor}(A, B)$  is an isomorphism

if there exists a two-sided inverse

$$g \in \text{Mor}(B, A) \quad \mathbb{1}_B = f \circ g; \quad g \circ f = \mathbb{1}_A$$

Examples of categories

- Sets and maps of sets  $\text{Set}$
- Abelian groups and group homomorphisms  $\text{Ab}$
- Modules over a ring  $A$  and homomorphisms of modules  $\text{Mod}_A$
- Topological spaces and continuous maps  $\text{Top}$
- Differentiable manifolds and differentiable maps  $\text{Diff}$
- Algebraic varieties and morphisms of varieties  $\text{Var}_k$
- Sheaves of abelian groups on a topological space  $X$   
+ morphisms of sheaves  $\text{Ab}(X)$

# Functors

Categories  $\mathcal{C}, \mathcal{D}$

$F: \mathcal{C} \rightarrow \mathcal{D}$  (covariant by default)

•  $F: \text{obj}(\mathcal{C}) \rightarrow \text{obj}(\mathcal{D})$

•  $\forall C_1, C_2 \in \mathcal{C} \quad \text{Mor}_{\mathcal{C}}(C_1, C_2) \rightarrow \text{Mor}_{\mathcal{D}}(F(C_1), F(C_2))$

preserving composition and identities

## Examples

• Forgetful functors, e.g.  $F: \text{Ab} \rightarrow \text{Sets}$ ,  $F(G) = G$   
("forget" that  $G$  is a group)

• Representable functors  $\forall A \in \mathcal{C}$  have

$h^A: \mathcal{C} \rightarrow \text{Sets}$   $h^A(B) = \text{Mor}(A, B)$

$h^A: \text{Mor}_{\mathcal{C}}(B, C) \rightarrow \text{Mor}_{\text{Sets}}(h^A(B), h^A(C))$ ,  $h^A(f): f \circ g$   
 $f: B \rightarrow C$   $g: A \rightarrow B$   $h^A(f): [A \rightarrow B] \rightarrow [A \rightarrow C]$

# Terminology

$F: \mathcal{C} \rightarrow \mathcal{D}$  is faithful if  $\forall c, c'$

$\text{Mor}_{\mathcal{C}}(c, c') \rightarrow \text{Mor}_{\mathcal{D}}(F(c), F(c'))$  is inj

full if surj

fully faithful isomorph

$\mathcal{C} \subset \mathcal{D}$  full subcategory:  $i: \mathcal{C} \rightarrow \mathcal{D}$   
full

Contravariant functor: reverse the arrows!

$$A \rightarrow B \rightarrow C$$

$\Downarrow$

$$F(A) \leftarrow F(B) \leftarrow F(C)$$

Examples

$X \rightsquigarrow H^i(X, \mathbb{Z})$  contravariant  $\text{Top}$  to  $\text{Ab}$

Representable functor (functor of points)

$A \in \mathcal{C}$   $h_A: \mathcal{C} \rightarrow \text{Set}$  contravariant

$$h_A(\mathcal{C}) = \text{Mor}(\mathcal{C}, A)$$

$$B \xrightarrow{f} C$$

$$h_A(\mathcal{C}) \rightarrow h_A(\mathcal{B})$$

$$[\mathcal{C} \xrightarrow{f} A] \rightarrow [\mathcal{B} \xrightarrow{f \circ g} A]$$

Opposite category  $\mathcal{C}^{\text{op}}$  to  $\mathcal{C}$

reverse the arrows

Then a contravariant functor from  $\mathcal{C}$  to  $\mathcal{D}$   
is identified with a covariant

$$F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$$

# Natural Transformation between two functors

Given categories  $\mathcal{C}, \mathcal{D}$

Functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$

A natural transformation  $F \xrightarrow{\eta} G$  is:

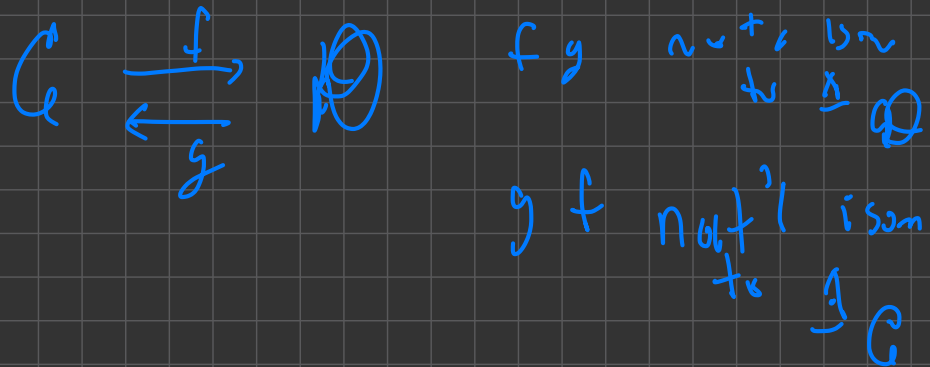
•  $\forall C \in \mathcal{C}$  a morphism  $\eta_C : F(C) \rightarrow G(C)$  s.t.

•  $\forall C \xrightarrow{f} C'$

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ F(f) \downarrow & \circlearrowleft & \downarrow G(f) \\ F(C') & \xrightarrow{\eta_{C'}} & G(C') \end{array}$$

Natural isomorphism: each  $\eta_C$  isomorphism

# Equivalence of categories



This is an equivalence relation on categories.

Equivalent categories may not be isomorphic!

Ex (C) Ex 1.2.1)  $Ob(\mathcal{V}) = \{k^n : n \in \mathbb{N}\}$   
+ linear trans

$Ob(\mathcal{Vect})$   $\ni$  finite dimensional vector spaces  $k$   
+ lin. trans

$\mathcal{V}$  and  $\mathcal{Vect}$  are equivalent, but not isomorphic

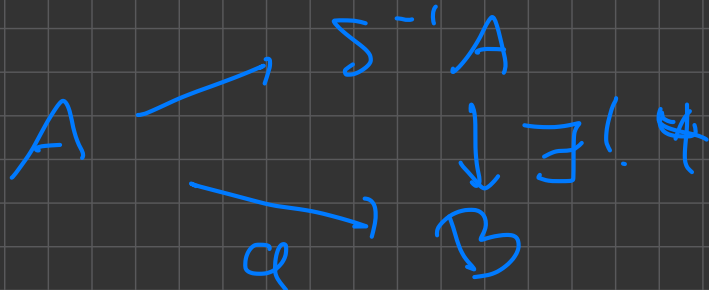


Useful concepts defined by universal properties.

• Localization A ring,  $S \subseteq A$  mult set

Consider  $A$ -algebras  $A \xrightarrow{q} B$  s.t.  $q(s)$  unit  $\forall s \in S$ .

$A \rightarrow S^{-1}A$  is universal for this property:



$$\frac{a}{s} \in S^{-1}A:$$

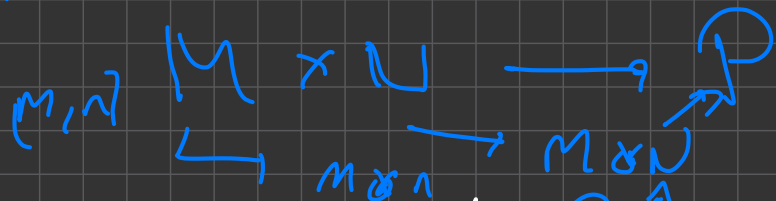
$$\psi\left(\frac{a}{s}\right) = q(s)^{-1} q(a)$$

• Tensor products of  $A$ -modules

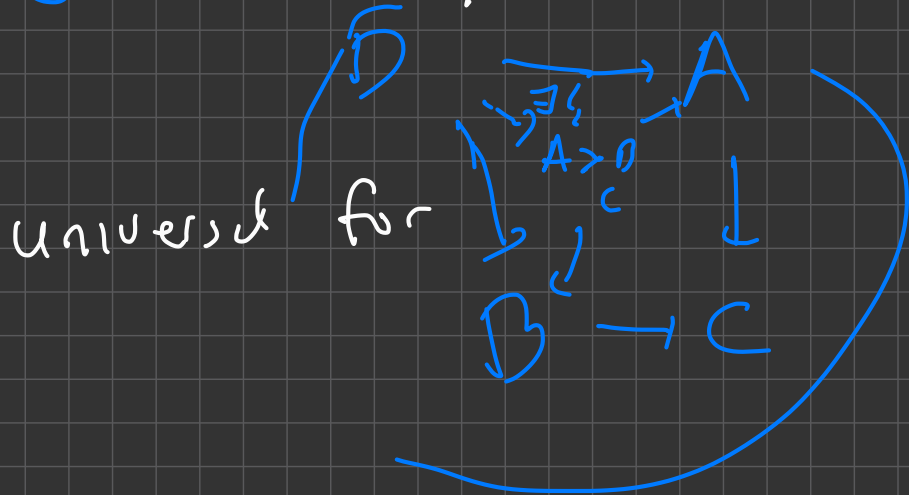
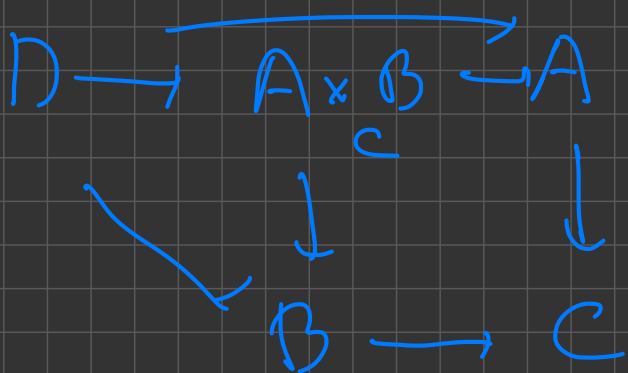
$$M \otimes_A N$$

universal for  $A$ -bilinear maps  
 $p \in \text{Hom}(M \times N, P)$

$M, N \in \text{Mod}_A$



• Fiber products (C arbitrary)



Yoneda's Lemma  $A, A' \in \text{Ob}(\mathcal{C})$ ;  $h_A, h_{A'}: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$

The natural transformations  $h_A$  to  $h_{A'}$  are in bijection with  $\text{Mor}_{\mathcal{C}}(A, A')$

PF (sketch) Given  $\alpha: h_A \rightarrow h_{A'}$  apply to  $A$

Given  $A \xrightarrow{f} A'$   $\mathbb{1}_A \in \text{Mor}(A, A) = h_A(A) \xrightarrow{\alpha} h_{A'}(A) = \underline{\underline{\text{Mor}(A, A)}}$

$$h_A \rightarrow h_{A'}$$

$$h_A(B) \rightarrow h_{A'}(B)$$

$$B \xrightarrow{g} A \xrightarrow{f} A' \quad B \xrightarrow{f \circ g} A'$$

Def A <sup>non empty</sup> category  $\mathcal{D}$  is filtered if

$$\forall x, y \in \mathcal{D} \quad \exists z \in \mathcal{D} \quad +$$

$$\forall x \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} y \quad \exists y \xrightarrow{h} z \quad \text{s.t.}$$

$$\begin{array}{c} x \\ \searrow \\ y \end{array} \xrightarrow{h} z$$

$$h \circ f = h \circ g$$

Exercise Suppose  $\mathcal{D}$  filtered  $F: \mathcal{D} \rightarrow \text{Sets}$  diagram

$$F(i) = S_i$$

$$\varinjlim_{\mathcal{D}} S_i = \left\{ (i, s_i) \in \coprod S_i \right\} /$$

$$\exists f: S_i \rightarrow S_k \quad g: S_j \rightarrow S_k \quad \text{in diagram}$$

$$(i, s_i) \sim (j, s_j) \quad \text{if}$$

$$f(a_i) = g(a_j) \in S_k$$

Compare to explicit formula for stalks of sheaves

# Adjoint functors

$F: \mathcal{C} \rightarrow \mathcal{D}$  left adjoint to  $G: \mathcal{D} \rightarrow \mathcal{C}$

(equiv:  $G$  right adjoint to  $F$ )

$\forall C \in \mathcal{C}, D \in \mathcal{D}$  have natural bijection

$$\text{Mor}_{\mathcal{D}}(F(C), D) \xrightarrow{\cong} \text{Mor}_{\mathcal{C}}(C, G(D))$$

Ex)

1.  $X$  top space  $\mathcal{C}$  presheaf of abelian gp on  $X$

$$\mathcal{D} = \text{Ab}(X)$$

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

$$\mathbb{F} \rightarrow \mathbb{F}^+$$

$$G: \mathcal{D} \rightarrow \mathcal{C} \text{ forgetful}$$

$$\mathbb{F} \text{ presheaf } (\mathbb{F} \in \mathcal{C})$$

$$g \text{ sheaf } (g \in \text{Ab}(X))$$

$$\text{Mor}_{\text{Ab}(X)}(\mathbb{F}^+, g) \longleftrightarrow \text{Mor}_{\mathcal{C}}(\mathbb{F}, g)$$

$$\begin{array}{ccc} \mathbb{F} & \rightarrow & g \\ \mathbb{F}^+ & \rightarrow & g \\ \mathbb{F} & \rightarrow & \mathbb{F}^+ \end{array}$$

is just the statement of the univ prop of the associated sheaf construction

2.  $f: X \rightarrow Y$  cont map of top. spaces

$$\text{Ab}(Y) \begin{array}{c} \xrightarrow{f^{-1}} \\ \xleftarrow{f_*} \end{array} \text{Ab}(X)$$

$f^{-1}$  left adjoint to  $f_*$

$$\mathcal{F} \in \text{Ab}(Y) \quad g \in \text{Ab}(X)$$

$$\text{Mor}_{\text{Ab}(X)}(f^{-1}\mathcal{F}, g) \cong \text{Mor}_{\text{Ab}(Y)}(\mathcal{F}, f_*g)$$

Sketch Given  $\alpha: \mathcal{F} \rightarrow f_*g$ , construct

$$\checkmark \quad \beta: f^{-1}\mathcal{F} \rightarrow g \quad \forall u \in X \text{ need}$$

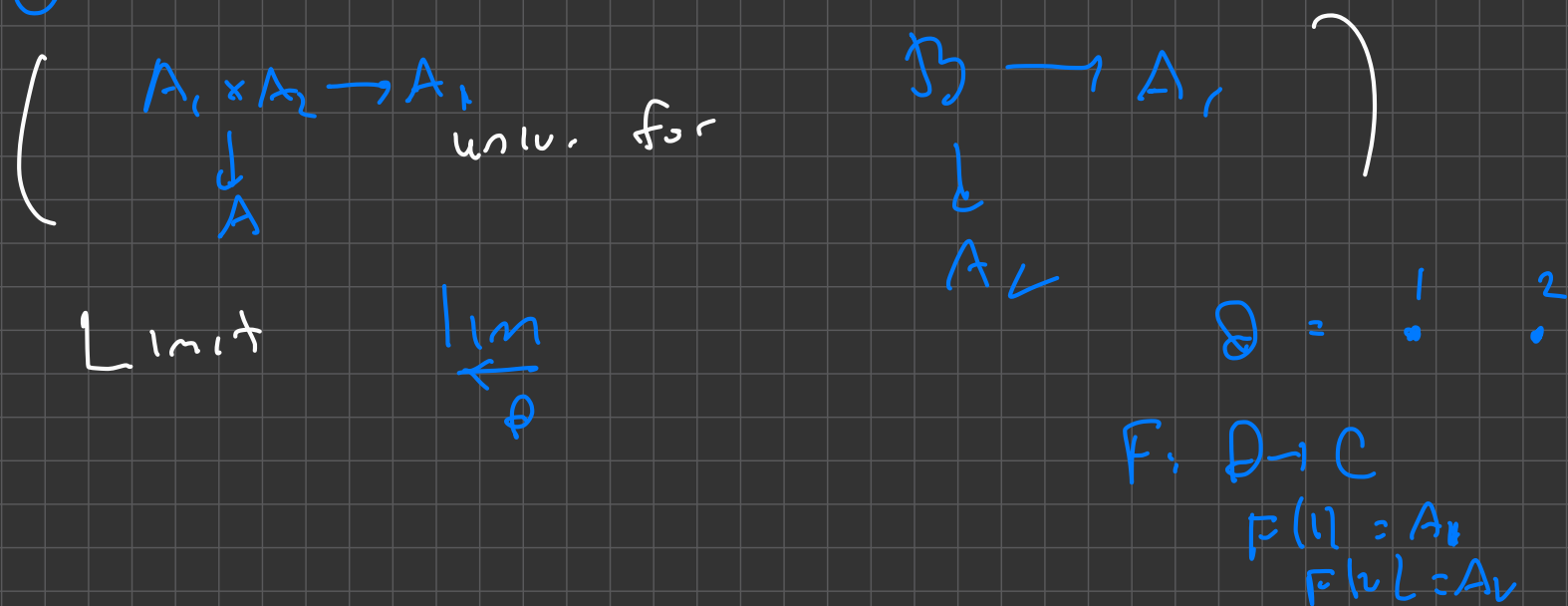
$$\forall v \in Y, \exists u \in X \text{ s.t. } f(u) = v \text{ here } \beta(f^{-1}(v)) \xrightarrow{\alpha(v)} (f_*g)(v) = g(f^{-1}(v))$$

By colimit get  $f^{-1}\mathcal{F}(u) \rightarrow g(u)$

# Abelian Categories

Def  $\mathcal{C}$  is an additive category if

- $\forall A, B \in \mathcal{C}$ ,  $\text{Mor}_{\mathcal{C}}(A, B)$  abelian gp
- $\forall A, B, C \in \mathcal{C}$  group homomorphisms  
 $\text{Mor}(B, C) \times \text{Mor}(A, B) \longrightarrow \text{Mor}(A, C)$
- $\mathcal{C}$  has a 0 object (initial + final)
- $\mathcal{C}$  has products





In additive category  $\mathcal{C}$

$$\text{Hom}_{\mathcal{C}}(A, B) := \text{Mor}_{\mathcal{C}}(A, B)$$

morphisms in  $\mathcal{C}$  called **homomorphisms**

$\mathcal{C}, \mathcal{D}$  additive categories

Def Additive functor  $F: \mathcal{C} \rightarrow \mathcal{D}$

- $F$  functor

- $\forall A, B \in \mathcal{C},$

$$F: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$$

homom of groups

$\mathcal{C}$  category with  $0$  object

Def. A kernel of  $f: A \rightarrow B$  is a morphism

$$i: C \rightarrow A \text{ s.t.}$$

$$f \circ i = 0$$

$i$  universal for this property

We already defined cokernels, a colimit.

Dually, a kernel is a limit of diagram

$$\begin{array}{c} A \\ \downarrow \\ 0 \rightarrow B \end{array}$$

$\mathcal{C}$  any category

Def  $A \xrightarrow{f} B$  is a monomorphism if

$\forall C, g_1, g_2: C \rightarrow A$  s.t.  $f \circ g_1 = f \circ g_2$  then  $g_1 = g_2$

In this case,  $A$  is a subobject of  $B$

Dual notion of epimorphism

$A \rightarrow B$  epimorphism, say  $B$  is a quotient object of  $A$

Def

An abelian category is an additive category s.t.

- Every homom. has a kernel + a cokernel
- Every monomorphism is the kernel of its cokernel
- Every epimorphism is the cokernel of its kernel

$$\begin{array}{c} A \xrightarrow{f} B \xrightarrow{g} \text{Coker}(f) \\ \cong \\ \text{ker}(g) \xrightarrow{g} \text{Coker}(f) \end{array}$$

## Examples

- $\text{Ab}$
- $\text{Mod}_A$
- $\text{Ab}(X)$

Let  $f: A \rightarrow B$  be a morphism in a category  $\mathcal{C}$

Def  $\text{Im} f := \text{ker}(\text{coker} f)$

Exercise If  $\text{Im} f$  exists,  $f$  factors as

$$A \xrightarrow{d} \text{Im} f \rightarrow B, \quad d \text{ epimorphism}$$

If  $\mathcal{C}$  is abelian

$$\text{Im} f = \text{coker}(\text{ker} f)$$

# Freyd - Mitchell Embedding Theorem

$\mathcal{C}$  abelian category. The  $\exists$  ring  $A$   
and exact (takes exact sequences to  
exact sequence)  
fully faithful

$$\mathcal{C} \longrightarrow \text{Mod}_A$$

embedding  $\mathcal{C}$  as a full subcategory.

abstract  
So many theorems about general abelian  
categories can be proven by diagram  
chasing, as if they were categories  
of modules!!