Vakil, Chapter 1

1.2 Categories and Functors

Categories : objects and morphisms

- Collection of objects $obj(\mathcal{C})$, $Ob(\mathcal{C})$, or just $\mathcal{C}$
- To any objects $A, B \in \mathcal{C}$, a set of morphisms $Mor(A,B)$
- Morphisms compose by a map of sets $\forall A,B,C \in \mathcal{C}$

$$Mor(B,C) \times Mor(A,B) \longrightarrow Mor(A,C)$$

Write $f \in Mor(A,B)$ as $f : A \to B$

Composition as $g \circ f : A \to C$ $\quad g : B \to C$

- $\forall A \in \mathcal{C}$ $\exists$ $1_A \in Mor(A,A)$ such that $\forall B \in \mathcal{C}, \forall f \in Mor(A,B)$ have $f = f \circ 1_A$; $f = 1_B \circ f$

A morphism $f \in \mathrm{Mor}(A, B)$ is an isomorphism if there exists a two-sided inverse $g \in \mathrm{Mor}(B, A)$ $1_B = f \circ g$; $g \circ f = 1_A$

Examples of categories

- Sets and maps of sets **Set**
- Abelian groups and group homomorphisms **Ab**
- Modules over a ring A and homomorphisms of modules $\mathbf{Mod}_A$
- Topological spaces and continuous maps **Top**
- Differentiable manifolds and differentiable maps **Diff**
- Algebraic varieties and morphism of varieties $\mathbf{Var}_k$
- Sheaves of abelian groups on a topological space X + morphisms of sheaves $\mathbf{Ab}(X)$

Functors

Categories $\mathcal{C}$, $\mathcal{D}$

$F : \mathcal{C} \longrightarrow \mathcal{D}$ (covariant by default)

- $F : \text{ob}(\mathcal{C}) \longrightarrow \text{ob}(\mathcal{D})$
- $\forall\, C_1, C_2 \in \mathcal{C}$ $\quad \text{Mor}_{\mathcal{C}}(C_1, C_2) \longrightarrow \text{Mor}_{\mathcal{D}}(F(C_1), F(C_2))$

preserving composition and identities.

Examples

- Forgetful functors, e.g. $F : \text{Ab} \to \text{Sets}$, $F(G) = G$

("forget" that G is a group)

- Representable functors $\forall\, A \in \mathcal{C}$ have

$$h^A : \mathcal{C} \longrightarrow \text{Sets} \quad h^A(B) = \text{Mor}(A, B)$$

$$h^A : \text{Mor}_{\mathcal{C}}(B, C) \longrightarrow \text{Mor}_{\text{Sets}}(h^A(B), h^A(C)), \quad h^A(f) : \quad f \circ g$$

$f : B \to C$ $\quad g : A \to B$ $\quad h^A(f)$ $\quad [A \to B] \to [A \to C]$

Terminology

$F : \mathcal{C} \to \mathcal{D}$ is faithful if $\forall\, c, c'$

$\mathrm{Mor}_{\mathcal{C}}(c, c') \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(c), F(c'))$ is inj

full if surj

fully faithful isomorph

$\mathcal{C} \subset \mathcal{D}$ full subcategory : $i : \mathcal{C} \to \mathcal{D}$

full

Contravariant functor : reverse the arrows!

$$A \longrightarrow B \longrightarrow C$$

$$\rightsquigarrow$$

$$F(A) \longleftarrow F(B) \longleftarrow F(C)$$

Examples

$X \rightsquigarrow H^i(X, \mathbb{Z})$ contravariant Top to Ab

Representable functor (functor of points)

$A \in \mathcal{C}$ $\quad h_A : \mathcal{C} \longrightarrow \mathrm{Sets}$ contravariant

$$h_A(C) = \mathrm{Mor}(C, A)$$

$B \xrightarrow{g} C$ $\qquad h_A(C) \to h_A(B)$

$$[C \xrightarrow{f} A] \to [B \xrightarrow{f \circ g} A]$$

Opposite category $\mathcal{C}^{op}$ to $\mathcal{C}$

reverse the arrows

Then a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is identified with a covariant

$$F : \mathcal{C}^{op} \longrightarrow \mathcal{D}$$

Natural Transform between two functors

Given categories $\mathcal{C}, \mathcal{D}$

Functors $F, G : \mathcal{C} \to \mathcal{D}$

A natural transformation $F \xrightarrow{m} G$ is:

- $\forall\, C \in \mathcal{C}$ a morphism $m_A : F(C) \to G(C)$ s.t.
- $\forall\, C \xrightarrow{f} C'$

$$\begin{tikzcd} F(C) \arrow[r, "m_C"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "m_{C'}"] & G(C') \end{tikzcd}$$

Natural isomorphism: each m_A isomorphism

Equivalence of categories

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$$

fg nat'l isom to $1_{\mathcal{D}}$

gf nat'l isom to $1_{\mathcal{C}}$

This is an equivalence relation on categories.

Equivalent categories may not be isomorphic!

Ex ([CV] Ex 1.2.D) $Ob(\mathcal{V}) = \{k^n : n \in \mathbb{N}\}$

+ linear transf

$Ob(Vect)$: $\exists$ finite dimensional vector spaces $/k$

+ lin. trans

$\mathcal{V}$ and $Vect$ are equivalent, but not isomorphic

Useful concepts defined by universal properties.

- **Localization** A ring, $S \subset A$ mult set

Consider A-algebras $A \xrightarrow{\varphi} B$ s.t. $\varphi(s)$ unit $\forall s \in S$.

$A \to S^{-1}A$ is universal for this property:

$$\begin{tikzcd} & S^{-1}A \arrow[d, "\exists!\,\psi"] \\ A \arrow[ur] \arrow[r, "\varphi"'] & B \end{tikzcd}$$

$\frac{a}{s} \in S^{-1}A$: $\qquad \psi\left(\frac{a}{s}\right) = \varphi(s)^{-1}\varphi(a)$

- Tensor products of A-modules

$M \otimes_A N$ universal for A-bilinear maps $\quad M, N \in \mathrm{Mod}_A$, $P \in A$-module

$$\begin{tikzcd} M \times N \arrow[r] \arrow[d] & P \\ M \otimes_A N \arrow[ur] & \end{tikzcd}$$

$(m,n) \mapsto m \otimes n$

- Fiber products ($\mathcal{C}$ arbitrary)

$$\begin{tikzcd} D \arrow[r] \arrow[rr, bend left] \arrow[dr, bend right] & A \times_C B \arrow[r] \arrow[d] & A \arrow[d] \\ & B \arrow[r] & C \end{tikzcd}$$

universal for

$$\begin{tikzcd} D \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times_C B \arrow[r] \arrow[d] & A \arrow[d] \\ & B \arrow[r] & C \end{tikzcd}$$

Yoneda's Lemma $A, A' \in Ob(\mathcal{C})$; $h_A, h_{A'} : \mathcal{C}^{op} \to Sets$

The natural transformations h_A to $h_{A'}$ are in bijection with $Mor_{\mathcal{C}}(A, A')$

Pf (sketch) Given $Q : h_A \to h_{A'}$ apply to A

$1_A \in Mor(A,A) = h_A(A) \to h_{A'}(A) = \underline{Mor(A,A')}$

Given

$A \xrightarrow{f} A'$

$h_A \to h_{A'}$

$h_A(B) \to h_{A'}(B)$

$B \xrightarrow{g} A \mapsto B \xrightarrow{f \circ g} A'$

Def A nonempty category $\mathcal{D}$ is **filtered** if

- $\forall\, x, y \in \mathcal{D}$ $\exists\, z \in \mathcal{D}$ + $\begin{array}{l} x \to z \\ y \to z \end{array}$
- $\forall\, x \underset{g}{\overset{f}{\rightrightarrows}} y$ $\exists\, y \xrightarrow{h} z$ s.t. $h \circ f = h \circ g$

Exercise Suppose $\mathcal{D}$ filtered $F : \mathcal{D} \to \mathrm{Sets}$ diagram

$F(i) = S_i$

$$\varinjlim_{\mathcal{D}} S_i = \Big\{ (i, s_i) \in \coprod_{i \in \mathcal{D}} S_i \Big\} \Big/$$

$(i, s_i) \sim (j, s_j)$ if

$\exists\, f : S_i \to S_k \quad g : S_j \to S_k$ in diagram

$f(a_i) = g(a_j) \in S_k$

Compare to explicit formula for stalks of sheaves

Adjoint functors

$F : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$

(equiv: G right adjoint to F

$\forall\ C \in \mathcal{C},\ D \in \mathcal{D}$ have natural bijection

$$\mathrm{Mor}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D))$$

Ex:

1. X top space $\mathcal{C}$ presheaves of abelian gps on X

$$\mathcal{D} = \mathcal{A}b(X)$$

$$F : \mathcal{C} \to \mathcal{D} \qquad \mathcal{F} \to \mathcal{F}^+$$

$$G : \mathcal{D} \to \mathcal{C} \quad \text{forgetful}$$

$\mathcal{F}$ presheaf $(\mathcal{F} \in \mathcal{C})$

$\mathcal{G}$ sheaf $(\mathcal{G} \in \mathcal{A}b(X))$

$$\mathrm{Mor}_{\mathcal{A}b(X)}(\underline{\mathcal{F}^+}, \mathcal{G}) \longleftrightarrow \mathrm{Mor}_{\underline{\mathcal{C}}}(\mathcal{F}, \mathcal{G})$$

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

is just the statement of the univ prop of the associated sheaf construction

2. $f: X \to Y$ cont map of top. spaces

$$Ab(Y) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} Ab(X)$$

f^{-1} left adjoint to f_*

$\mathcal{F} \in Ab(Y) \qquad g \in Ab(X)$

$$Mor_{Ab(X)}(f^{-1}\mathcal{F}, g) \longleftrightarrow Mor_{Ab(Y)}(\mathcal{F}, f_* g)$$

Sketch Given $\alpha : \mathcal{F} \to f_* g$, construct

$f^{-1}\mathcal{F} \to g$. $\forall U \subset X$ need

$\forall V \subset Y$, $f(U) \subset Y$ here $\mathcal{F}(V) \xrightarrow{\alpha(V)} (f_* g)(V) = g(f^{-1}(V))$

$U \subset f^{-1}(V)$

By colimit get $f^{-1}\mathcal{F}(U) \to g(U)$

Abelian Categories

Def $\mathcal{C}$ is an *additive category* if

- $\forall A, B \in \mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(A, B)$ abelian gp
- $\forall A, B, C \in \mathcal{C}$ group homomorphism
$$\mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \longrightarrow \mathrm{Mor}(A, C)$$
- $\mathcal{C}$ has a 0 object (initial + final)
- $\mathcal{C}$ has products

($A_1 \times A_2 \to A_1$, $\downarrow$ A univ. for $B \to A_1$, $\downarrow$ A_2)

Limit $\varprojlim_{D}$

$D = \overset{1}{\bullet} \quad \overset{2}{\bullet}$

$F: D \to C$

$F(1) = A_1$

$F(2) = A_2$

In additive category $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(A,B) := \mathrm{Mor}_{\mathcal{C}}(A,B)$$

morphisms in $\mathcal{C}$ called homomorphisms

$\mathcal{C}, \mathcal{D}$ additive categories

Def Additive functor $F : \mathcal{C} \to \mathcal{D}$

- F functor
- $\forall A, B \in \mathcal{C}$,

$$F : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

homom of groups

$\mathcal{C}$ category with 0 object

Def A **kernel** of $f : A \to B$ is a morphism $i : C \to A$ s.t.

- $f \circ i = 0$
- i universal for this property

We already defined cokernels, a colimit.
Dually, a kernel is a limit of diagram

$$\begin{tikzcd} & A \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}$$

$\mathcal{C}$ any category

Def $A \xrightarrow{f} B$ is a monomorphism if

$\forall\ C,\quad g_1, g_2 : C \longrightarrow A$ s.t. $f \circ g_1 = f \circ g_2$

then $g_1 = g_2$

In this case, A is a subobject of B

Dual notion of epimorphism

$A \longrightarrow B$ epimorphism, say B is a quotient object of A

Def An **abelian category** is an additive category s.t.

- Every homom. has a kernel + a cokernel
- Every monomorphism is the kernel of its cokernel
- Every epimorphism is the cokernel of its kernel

$$\begin{tikzcd} & A \arrow[r, hook, "f"] \arrow[dl, phantom, "\cong" description] & B \arrow[r, "g"] & \mathrm{Coker}(f) \\ \ker(g) \arrow[r, hook] & B \arrow[r, "g"] & \mathrm{Coker}(f) & \end{tikzcd}$$

Examples

- Ab
- Mod_A
- $Ab(X)$

Let $f: A \to B$ be a morphism in a category $\mathcal{C}$

Def $\operatorname{Im} f := \ker(\operatorname{coker} f)$

Exercise If $\operatorname{Im} f$ exists, f factors as

$$A \xrightarrow{\alpha} \operatorname{Im} f \longrightarrow B, \quad \alpha \text{ epimorphism}$$

If $\mathcal{C}$ is abelian

$$\operatorname{Im} f = \operatorname{coker}(\ker f)$$

Freyd - Mitchell Embedding Theorem

$\mathcal{C}$ abelian category. The $\exists$ ring A

and exact (takes exact sequences to exact sequence)

fully faithful

$$\mathcal{C} \longrightarrow Mod_A$$

embedding $\mathcal{C}$ as a full subcategory.

So many abstract theorems about general abelian categories can be proven by diagram chasing, as if they were categories of modules!!