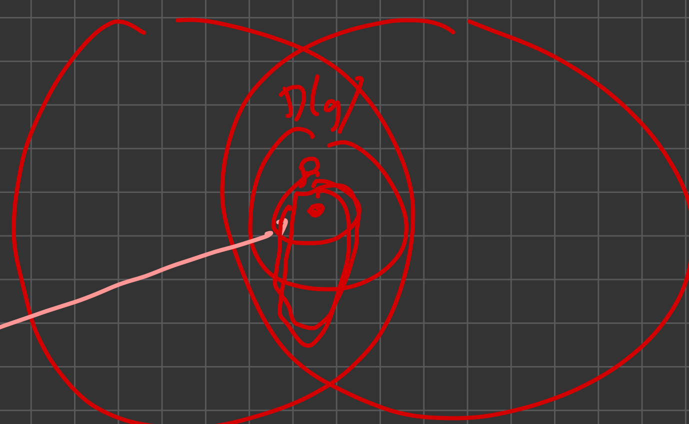
Completion of [H] Exercise II.2.4

([V] Prop 5.3.1) Suppose $\operatorname{Spec} A$, $\operatorname{Spec} B$ open subschemes of X. Then $\operatorname{Spec} A \cap \operatorname{Spec} B$ is a union of open subsets simultaneously of form $D(f) = \operatorname{Spec} A_f$ $(f \in A)$ $= \underset{\underset{D(g)}{\shortparallel}}{\operatorname{Spec} B_g}$ $(g \in B)$

Pf

$\operatorname{Spec} A_f = D(f)$



Spec A   Spec B

$g \in B \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$

restricts to

$g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$

$V((g)) \cap \operatorname{Spec} A_f = V((g')) \cap \operatorname{Spec} A_f \implies$

$\operatorname{Spec} B_g = \operatorname{Spec} A_f - \{\mathfrak{p} \subset A_f : g' \in \mathfrak{p}\} = \operatorname{Spec}[(A_f)_{g'}]$

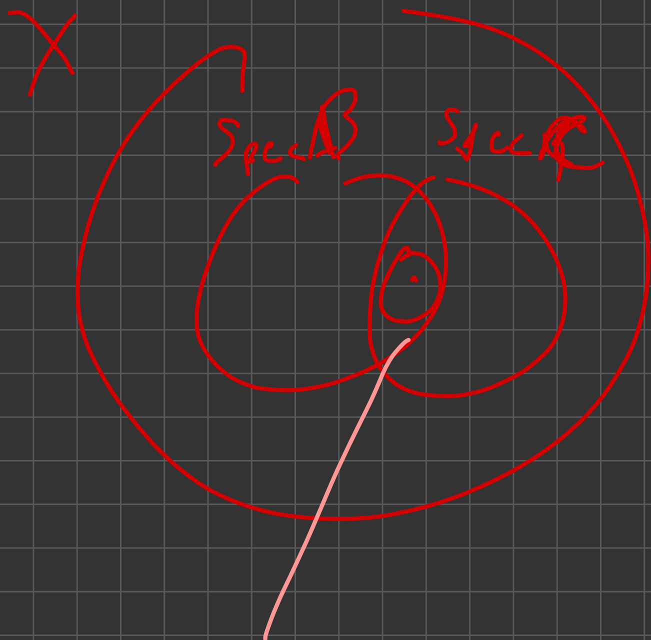
Write $\left(g' = \frac{g''}{f^n}\right) \implies \operatorname{Spec} B_g = \operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$

$$X \longrightarrow \operatorname{Spec} A$$

$$A \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(\operatorname{Spec} B, \mathcal{O}_X) = B \longrightarrow B_f$$

$$\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(\operatorname{Spec} C, \mathcal{O}_X) = C \longrightarrow C_g$$

$$B_f \cong C_g$$



$$\begin{tikzcd}
\operatorname{Spec} B_f \arrow[r] \arrow[d, equal] & \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \\
\operatorname{Spec} C_g \arrow[r] & \operatorname{Spec} C \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

So mappings glue.

$$\operatorname{Spec} B_f = \operatorname{Spec} C_g$$

$f \in B \qquad g \in C$

Def A scheme X is **connected** if $Sp(X)$ is connected
irreducible if $Sp(X)$ is irred.

Def A scheme X is **reduced** if $\forall$ open $U \subset X$,
$\mathcal{O}_X(U)$ has no nilpotent elements

Def A scheme X is **integral** if $\forall$ open $U \subset X$,
$\mathcal{O}_X(U)$ is an integral domain

Lemma X is reduced $\iff$ $\forall p \in X$ $\mathcal{O}_{X,p}$ has no nilpotents

Prop X is integral $\iff$ X is reduced and irreducible

Examples of schemes

Already affine schemes give interesting examples

$k = \bar{k}$ $\quad \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n] =: \operatorname{Spec} R$ reduced irreducible integral

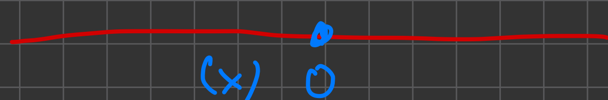
$I \subset R$ any ideal

$R \to R/I$ determines $\operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$

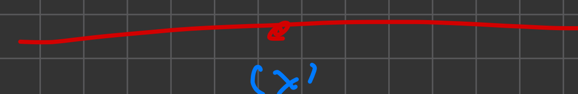
$$sp(\operatorname{Spec} R/I) = V(I)$$

closed subscheme

Compare $\operatorname{Spec} k[x]/(x)$, $\operatorname{Spec} k[x]/(x^2)$



reduced irreducible



non-reduced

Global sections $= k[x]/x^2$

x nilpotent

$\operatorname{Spec} k[x,y]/(xy)$

$x \cdot y = 0$ $\qquad \| \qquad V(x) \cup V(y)$

reducible

reduced

not integral

In this course, a top. space is *quasi-compact* if every open cover has a finite subcover.

Lemma For any ring A, $X = \operatorname{Spec} A$ is quasi-compact.

Pf: Suffices to prove for $\underset{\ni \, \mathfrak{m}}{X} = \bigcup_{i \in I} U_i = \bigcup_{i \in I} D(a_i)$, $a_i \in A$. Suppose $(a_i)_{i \in I} \neq A$.

Choose maximal $\mathfrak{m} \supseteq (a_i)_{i \in I}$

Then $\forall i$: $\mathfrak{m} \ni a_i \implies \mathfrak{m} \notin D(a_i)$ contradiction

So $(a_i) = (1)$ $\qquad 1 = \sum_{\substack{\text{finite} \\ i \in J}} c_i a_i$ $\qquad |J| < \infty$ $\quad J \subset I$

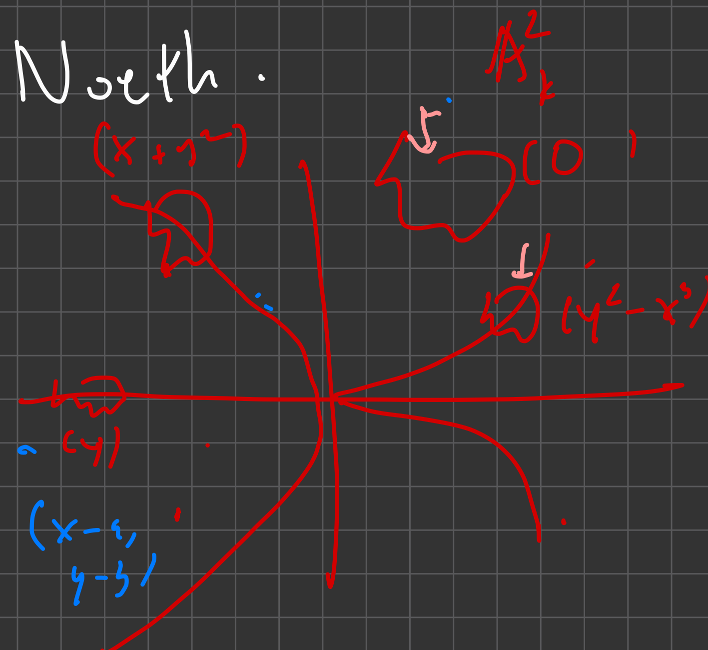
$\operatorname{Spec} A = \bigcup_{i \in J} D(a_i)$

Def X is **locally Noetherian** if it can be covered by open affines $\operatorname{Spec} A_i$ with each A_i Noetherian.

X is **Noetherian** if it is locally Noeth + quasi-compact

Equivalently $X = \bigcup_{i=1}^{n} \operatorname{Spec}(A_i)$ A_i Noeth

Ex $X = \operatorname{Spec} k[x,y]/(y^2-x^3)$ is Noeth.



$$k[x,y] \longrightarrow k[x,y]/(y^2-x^3)$$

$$X \longrightarrow \mathbb{A}^2_k$$

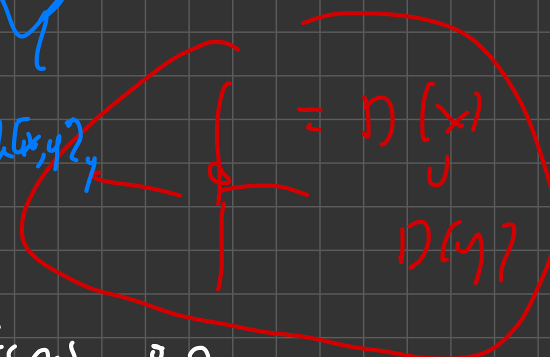
$$X = \{(y^2-x^3), (0)\}$$

Def An open subscheme of a scheme X is a scheme $\mathcal{U}$, $sp(\mathcal{U}) \subset sp(X)$ open,

$$\mathcal{O}_{\mathcal{U}} \cong \mathcal{O}_X|_{\mathcal{U}}$$

Open immersion $f : X \to Y$ inducing isom of X with an open subscheme of Y

$$D(x) \cup D(y) = \operatorname{Spec} k[x,y]_x \cup \operatorname{Spec} k[x,y]_y$$



Ex $X = \mathbb{A}^2_k - 0 \subset \mathbb{A}^2_k$ Noetherian

$\Gamma(\mathbb{A}^2_k - 0, \mathcal{O})$ = global regular fcns on variety $\mathbb{A}^2 - 0 = k[x,y]$

$\mathbb{A}^2 - 0$ not affine

$$\mathcal{O}_X(\mathbb{A}^2 - 0) = \mathcal{O}_X(D(x)) \cap \mathcal{O}_X(D(y)) \subset K(X) = k(x,y)$$

$$k[x,y,\tfrac{1}{x}] = k[x,y]_x \cap k[y,x,\tfrac{1}{y}] = k[x,y]$$

Def A **closed subscheme** of X is a scheme Y and morphism $\iota : Y \to X$ s.t. $sp(Y)$ closed subset of $sp(X)$ and

$$\iota^\# : \mathcal{O}_X \to \iota_* \mathcal{O}_Y$$ is surjection

A **closed immersion** is a morphism $f : Y \to X$ inducing isom of Y with a closed subscheme of X

Example A ring $\mathfrak{a} \subset A$ ideal

$$\operatorname{Spec} A/\mathfrak{a} \xrightarrow{f} \operatorname{Spec} A$$

induced from $A \to A/\mathfrak{a}$

closed immersion $f : sp(A/\mathfrak{a}) \underset{\text{homeo}}{\cong} V(\mathfrak{a})$

$$f^\# : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} A/\mathfrak{a}} \qquad \mathfrak{p} \supset \mathfrak{a}$$

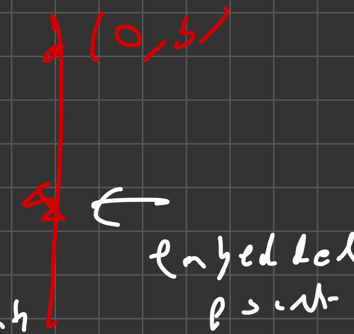
on stalk $A_\mathfrak{p} \twoheadrightarrow (A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}}$

non reduced

Ex $A = k[x,y]$ $\quad \mathfrak{a} = (x^2, xy)$

$\mathfrak{p} \subset k[x,y]/(x^2,xy) =: R$ Just look at $\mathfrak{p}$ maximal

$\mathfrak{p} = (x, y-b)$



$R_{\mathfrak{p}}$ $b=0$ $x^2=0$ in $R_{\mathfrak{p}}$ has nilpotent

$b \neq 0$ $x^2=0$ in $R_{\mathfrak{p}}$ has no nilpotent

y is a unit in $R_{\mathfrak{p}}$

$xy=0 \Rightarrow x=0$

$F^{-1}(0)=(x)$

$$k[x,y]/(x^2,xy) \xrightarrow{F} k[x,y]/(x) \quad (0)$$

$$\operatorname{Spec} k[x,y]/(x) \hookrightarrow k[x,y]/(x^2,xy)$$

Prop Fiber products exist in category of schemes

$$
\begin{tikzcd}
\operatorname{Spec} D \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} C \arrow[r] \arrow[d] & \operatorname{Spec} C \arrow[d] \\
& \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B
\end{tikzcd}
$$

Basic idea

reduce to affines and glue

$$
\begin{tikzcd}
D & & \\
& ? \arrow[ul, dashed] & C \arrow[l] \arrow[ull, bend right] \\
& A \arrow[u] \arrow[uul, bend left] & B \arrow[l] \arrow[u]
\end{tikzcd}
$$

$? = A \otimes_B C$

$$\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} C \cong \operatorname{Spec}(A \otimes_B C)$$

Fiber of a morphism $f: X \to Y$ over $y \in Y$

$k(y) = \mathcal{O}_{Y,y}/m_y$. Pick nbhd $y \in \operatorname{Spec} A \subset Y$

$y \leftrightarrow \mathfrak{p} \subset A$

$k(y) = A_{\mathfrak{p}}/m_y A_{\mathfrak{p}}$

$A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/m_y A_{\mathfrak{p}} \cong k(y)$ induces

$\operatorname{Spec} k(y) \hookrightarrow \operatorname{Spec}(A) \subset Y$

Check $\operatorname{Spec} k(y) \to Y$ indep of nbhd $\operatorname{Spec} A$

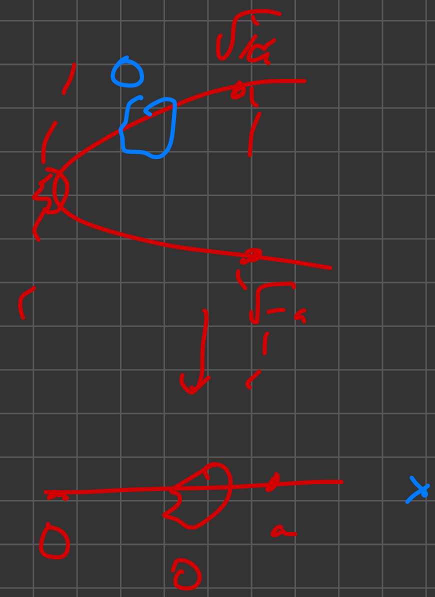
Def Fiber $X_y = X \times_Y \operatorname{Spec} k(y)$

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r] & Y \end{tikzcd}$$

Remark $\operatorname{sp}(X_y)$ homeo to $f^{-1}(y) \subset X$

Ex: $f: X = \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$

$$y \longmapsto y^2 = x$$



$\mathfrak{p} = (x-a) \in \mathbb{A}^1_k$

$$X_a = \mathrm{Spec}\left(k[y] \otimes_{k[x]} k[x]/(x-a)\right)$$

$$\cong \mathrm{Spec}\, k[y]/(y^2-a)$$

Generic fiber $(0) \in \mathbb{A}^1_k$

$$X_{(0)} \cong \mathrm{Spec}\left(k[y] \otimes_{k[x]} k(x)\right)$$

$$\cong \mathrm{Spec}\, k(x)[y]/(y^2 -)$$

$$X_{(x)} = \mathrm{Spec}\, k[y]/(y^2)$$