Ex $\operatorname{Spec} k[x,y]/(x^2, xy, y^2)$

$m = (x,y)$

$$\operatorname{Spec} k \overset{\sim}{\hookrightarrow} \operatorname{Spec} k[x,y]/m \hookrightarrow \operatorname{Spec} k[x,y]/m^2 \hookrightarrow \operatorname{Spec} k[x,y]/m^3 \cdots$$

$$k[x,y]/m^2 \cong \left(k[x,y]/m^3\right)/\left(m^2/m^3\right)$$

Ex $\mathbb{A}^\infty_k = \operatorname{Spec} k[x_1, x_2, x_3, \ldots]$

Not Noetherian or even locally Noeth

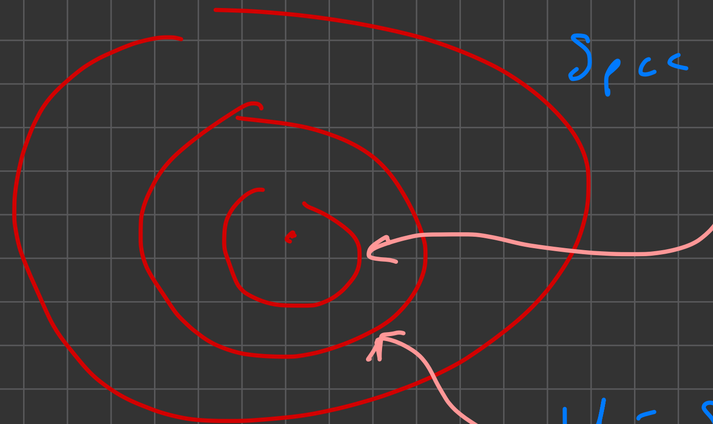
Prop X is locally Noetherian if and only if $\forall$ $U = \operatorname{Spec} A \subset X$ open affine, A is a Noetherian ring. In particular, $\operatorname{Spec} A$ is a Noetherian scheme if and only if A is Noetherian

Proof $\Leftarrow$ obvious

$\Rightarrow$ Suppose X loc. Noeth, $\operatorname{Spec} A \subset X$

$X = \bigcup \operatorname{Spec} B_i$, B_i Noeth

$\therefore$ $\operatorname{Spec} A$ can be covered by spectra of Noeth rings.



Spec $A = X$

$V = D(f)$, $f \in A$

B Noeth

$U = \operatorname{Spec} B$

$\operatorname{Spec} B \to \operatorname{Spec} A \dashrightarrow \overset{f\in}{A} \to B$, $\bar{f} \in B$ image of f

$A_f \cong B_{\bar{f}}$ Noeth

$\therefore$ X can be covered with opens $\operatorname{Spec} A_f$ with A_f

X q.e. $\Longrightarrow$ can cover with finitely many

$\operatorname{Spec} A_{f_1}, \dots, \operatorname{Spec} A_{f_r}$

$X = D(f_1) \cup \dots \cup D(f_r)$

$(f_1, \dots, f_r) = (1)$ $\quad A_{f_i}$ Noeth

Lemma $\quad \mathfrak{a} \subset A$ ideal $\quad \varphi_i : A \to A_{f_i}$, then

$$\mathfrak{a} = \bigcap_{i=1}^{r} \varphi_i^{-1}\big(\underbrace{\varphi_i(\mathfrak{a}) \cdot A_{f_i}}_{\mathfrak{a}_i \subset A_{f_i}}\big)$$

Pf: $\subseteq$ clear

If $b \in \bigcap_{i=1}^{n} \varphi_i^{-1}\big(\underbrace{\varphi_i(\mathfrak{a}) \cdot A_{f_i}}_{\mathfrak{a}_i}\big)$, $\varphi_i(b) = \frac{b}{1} = \frac{a_i}{f_i^{n_i}}$ $\quad a_i \in \mathfrak{a}$

WLOG $\quad n = n_1 = \dots = n_r$

$\exists m_i \quad f_i^{m_i}(f_i^n b - a_i) = 0$

WLOG $\quad m = m_1 = \dots = m_r$

$\Rightarrow f_i^{m+n} b = f_i^m a_i \in \mathfrak{a}$

Put $N = nrm$, have $f_i^N b \in \mathfrak{a}$, $i = 1, \dots, r$

$$(1) = (f_1, \dots, f_r) \iff D(f_i) \text{ cover } X$$

$$(1) = (f_1^N, \dots, f_r^N) \iff D(f_i^N)$$

$$1 = \sum c_i f_i^N \qquad c_i \in A$$

$$\Rightarrow b = \sum c_i \underbrace{f_i^N b} \in \mathfrak{a}$$

∎

Now prove A Noeth. Let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$

$$\Rightarrow \alpha_i(\mathfrak{a}_1)\cdot A_{f_i} \subset \alpha_i(\mathfrak{a}_2)\cdot A_{f_i} \subset \alpha_i(\mathfrak{a}_3)\cdot A_{f_i} \subset \cdots$$

Eventually for $j \geq M \gg 0$ $\quad \alpha_i(\mathfrak{a}_j)\cdot A_{f_i} = \alpha_i(\mathfrak{a}_{j+1})\cdot A_{f_i} = \cdots$

WLOG same M for all i

$$\Rightarrow \mathfrak{a}_j = \mathfrak{a}_{j+1} = \cdots$$

∎

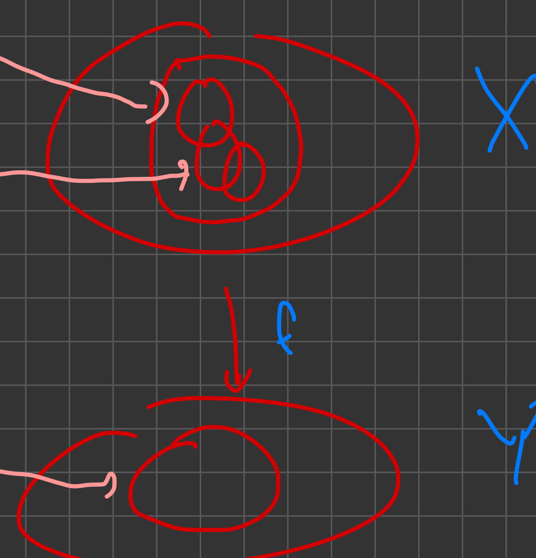
Def $f: X \to Y$ **locally of finite type** if Y can be covered by open affines $V_i = \operatorname{Spec} B_i$ s.t. $f^{-1}(V_i)$ covered by open affines $U_{ij} = \operatorname{Spec} A_{ij}$ s.t.

A_{ij} is a finitely generated B_i-algebra

f is of **finite type** if in addition each $f^{-1}(V_i)$ can be covered by finitely many U_{ij}

$$f^{-1}(V_i)$$

$$\begin{array}{ccc} A_{ij} & & U_{ij} = \operatorname{Spec} A_{ij} \\ \uparrow & & \downarrow \\ B_i & & \operatorname{Spec} B_i = V_i \end{array}$$



Def $f: X \to Y$ is a **finite morphism** if $\exists$ covering of Y by open affines $V_i = \operatorname{Spec} B_i$ s.t.

- $f^{-1}(V_i)$ is affine, $\cong \operatorname{Spec} A_i$
- A_i is finite as a B_i-module

Remark: like the locally Noeth case, these definitions are equivalent to requiring that these properties hold for all open affine subsets. See [H] Ex II 2.3.1 – II 2.3.4

Exs: $X = \mathbb{A}^0_k \sqcup \mathbb{A}^1_k \sqcup \mathbb{A}^2_k \sqcup \cdots$

loc. Noeth, not Noeth

$X \longrightarrow \operatorname{Spec} k$

locally finite type, not finite type

$\operatorname{Spec} k = \mathbb{A}^n_k \longrightarrow k$ abuse of notation

$k \longrightarrow k[x_1, \ldots, x_n]$

$\mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$

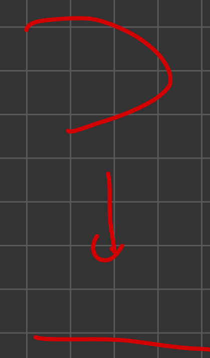
$x \longmapsto y = x^2$

finite

finite $k[y]$-mod

$k[y] \longrightarrow k[x]$

$y \longmapsto x^2$



$k[x] = k[y]\cdot 1 + k[y]x$

aff

$\mathbb{A}^2_k - 0 \overset{i}{\hookrightarrow} \mathbb{A}^2$

finite type, not finite

$i^{-1}(\mathbb{A}^2) = \mathbb{A}^2 - 0$ not aff

$\{D(x) \cup D(y)\} = i^{-1}(\mathbb{A}^2)$

$i^{-1}(D(x)) = D(x)$.

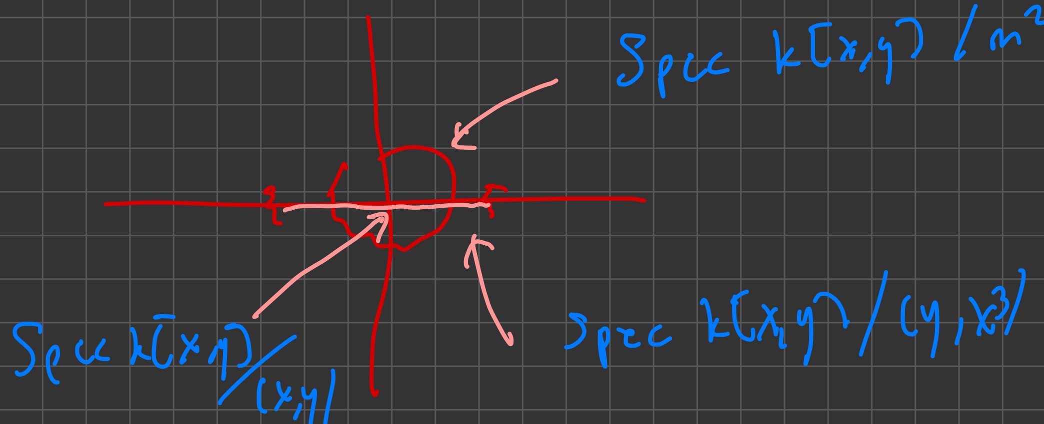
$\operatorname{Spec} k[x,y]_{(x)}$

$k[x,y]_{(x)} \xrightarrow{\cong} k[x,y]_{(x)}$ f.g.

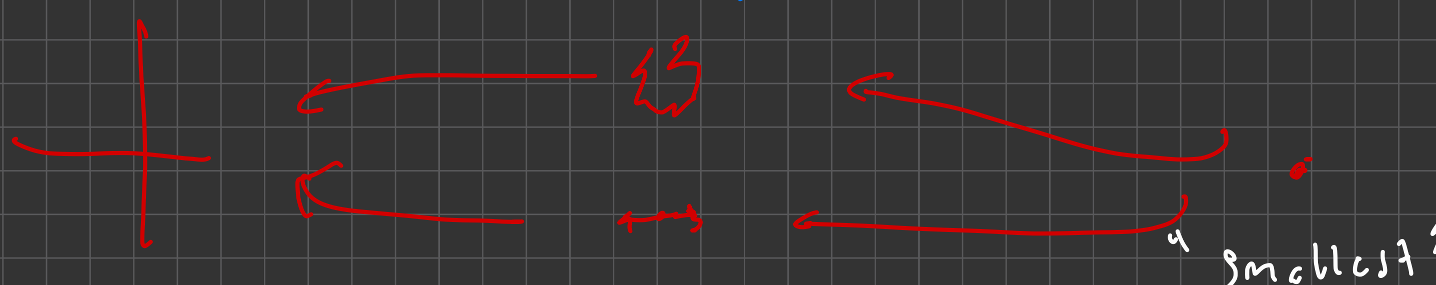
More about closed subscheme

X scheme $Y \subset X$ closed. Usually Y has many closed subscheme structures

Ex $X = \mathbb{A}^2_k$ $Y = \{0\}$ $m = (x,y)$



$$\begin{tikzcd}
 & k[x,y]/(x^2,xy,y^2) \arrow[dr] & \\
k[x,y] \arrow[ur] \arrow[dr] & & k[x,y]/(x,y) \\
 & k[x,y]/(y,x^3) \arrow[ur] &
\end{tikzcd}$$



In general Y has a canonical "smallest" subscheme structure, the reduced induced closed subscheme structure.

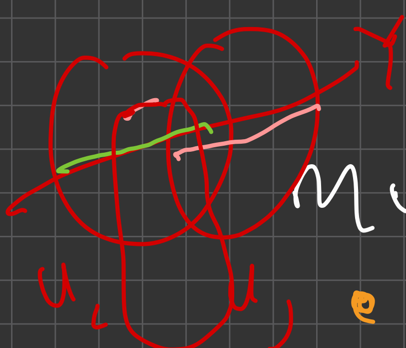
- $X = \operatorname{Spec} A$ affine, $Y \subset X$ closed

$$\mathfrak{a} := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} \qquad (= \sqrt{I} \text{ if } Y = V(I))$$

Reduced induced structure defined by $\mathfrak{a}$

- Cover X by open affines $U_i = \operatorname{Spec} B_i$

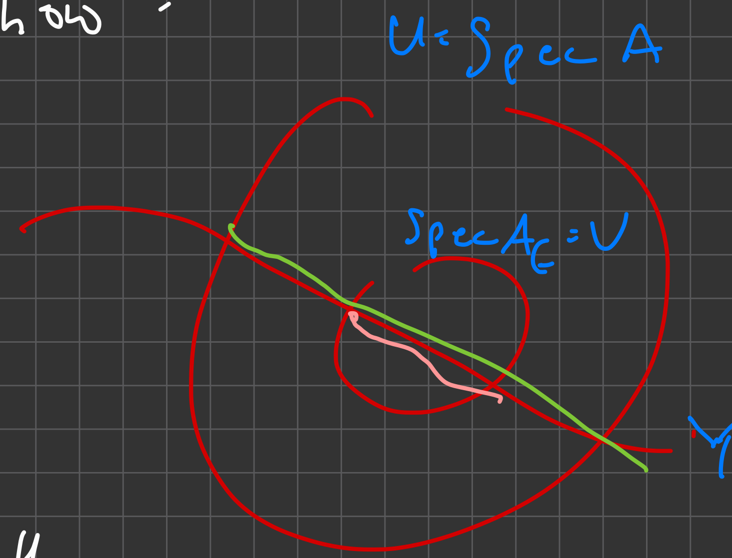
give $Y \cap U_i \subset U_i$ reduced induced structure



Must show

- Reduced induceds on $Y \cap U_i \subset U_i$, $Y \cap U_j \subset U_j$ are isomorphic on $Y \cap U_i \cap U_j$
- These isomorphisms are compatible on triple intersections $Y \cap U_i \cap U_j \cap U_k$

By now-standard techniques for comparing open affines, suffices to show:



$U = \operatorname{Spec} A$, $f \in A$,

$V = \operatorname{Spec} A_f$

reduced induced structure on $Y \cap U$ from A agrees with reduced induced on $Y \cap V$ from A_f

Need only compare ideals

$$\mathfrak{a} = \bigcap_{A \supset \mathfrak{p} \in Y \cap U} \mathfrak{p} \Longrightarrow \mathfrak{a} A_f = \bigcap_{A_f \supset \mathfrak{q} \in Y \cap V} \mathfrak{q}$$

$$\updownarrow$$

$$A \supset \mathfrak{p} \not\ni f$$

$$\left(\operatorname{Spec} A/\mathfrak{a}\right)\Big|_{\operatorname{Spec} A_f} \cong \operatorname{Spec} A_f/\mathfrak{a}A_f$$

Dimension

X scheme $\quad \dim X$ is its dimension as a top space

$$= \sup \{ n \mid \exists\, X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \}$$

X_i distinct irred closed

In particular $\dim \operatorname{Spec} A = \dim A$ (Krull dimension)

If $Z \subset X$ irred + closed

$$\operatorname{codim}(Z, X) = \sup \{ n \mid Z = Z_0 \subset Z_1 \cdots \subset Z_n \}$$

If $Y \subset X$ closed

$$\operatorname{codim}(Y, X) = \inf_{\substack{Z \subset Y \\ \text{irred}}} \{ \operatorname{codim}(Z, X) \}$$

$\cdot \subset$ —— $\subset \mathbb{A}^2$

$(x,y) \supset (y) \supset 0$

$l \subset \mathbb{A}^2$

In $\mathbb{A}^2_k$ $\quad \operatorname{codim}(\{0\}, \mathbb{A}^2_k) = 2 \quad \operatorname{codim}(l, \mathbb{A}^2_k) = 1 \quad \operatorname{codim}(Y, \mathbb{A}^2_k) = 1 = \inf(2, 1, 2 \ldots)$

[H] II.4 Separated and Proper Morphisms

$f : X \longrightarrow Y$

diagonal morphism $\Delta : X \longrightarrow X \times_Y X$

characterized by $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{id}_X$

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Def: f is separated if

Δ is a closed immersion

Prop Morphisms of affine schemes are separated

Pf: $f: \underset{\overset{\shortparallel}{X}}{\operatorname{Spec} B} \to \underset{\overset{\shortparallel}{Y}}{\operatorname{Spec} A}$

$$X \to X \times_Y X = \operatorname{Spec}(B \otimes_A B)$$

$$0 \leftarrow B \xleftarrow{m} B \otimes_A B \qquad B = B \otimes_A B / I_{\text{kernel}}$$

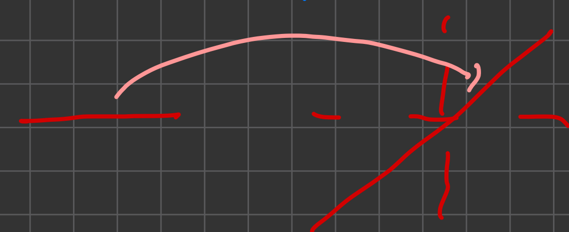
$$b_1 b_2 \leftarrow\!\shortmid\, b_1 \otimes b_2$$

$$b = m(b \otimes 1)$$

$\square$

Ex: $\mathbb{A}^1_k \to \operatorname{Spec} k \qquad \mathbb{A}^1_k \times_k \mathbb{A}^1_k \simeq \mathbb{A}^2_k$

$$\Delta: \mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$$



$\operatorname{Spec} k[x,y]/(y-x)$

$$k[x] \leftarrow k[x] \otimes k[y]$$

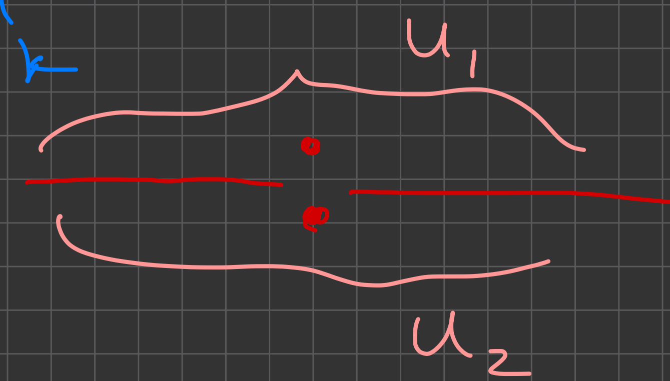
$$x \leftarrow\!\shortmid\, x$$

$$x \leftarrow\!\shortmid\, y$$

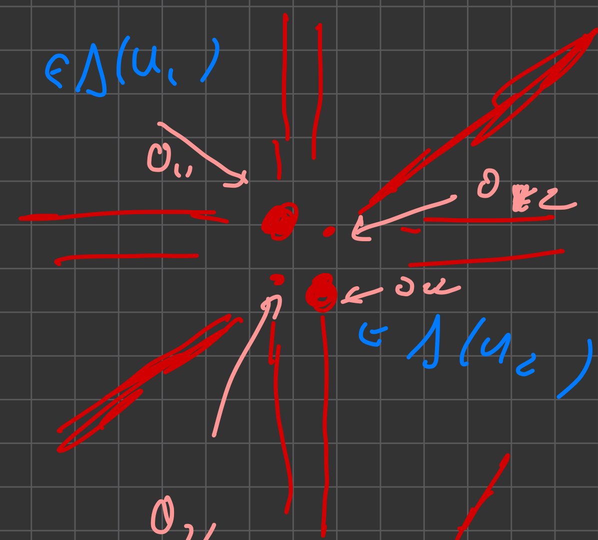
Ex $\mathbb{A}^1_k$ with origin doubled is not separated over k

$X = U_1 \cup U_2$ $\quad U_i \cong \mathbb{A}^1_k$

$U_i - \{0\}$ glued by identity of



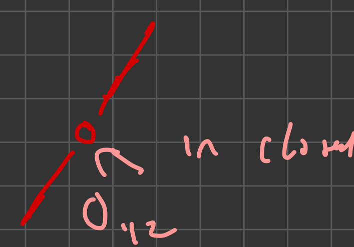
$X = \bigcup_{i,j=1,2} U_i \times U_j$ glued by identity of



$\overline{\Delta(X)} \ni O_{12} \notin \Delta(X)$

$\cap$

$U_1 \times U_2 \cong \mathbb{A}^2$ $\quad \Delta(X) \cap U_1 \times U_2 =$



$\Delta(X)$ not closed

Δ not closed immersion

This example illustrates:

$f: X \to Y$ separated iff $\Delta(X) \subset X \times_Y X$ closed

PF: $\Longrightarrow$ clear

$\Longleftarrow$ must check

- $X \xrightarrow{\Delta} \Delta(X) \subset X \times X$ homeo
- $\Delta^\#: \mathcal{O}_{X \times X} \longrightarrow \Delta_* \mathcal{O}_X$ surj

$X \to \Delta(X) \xrightarrow{p_1} X$

For 1^{st} point, have cont. inverse

Surjectivity is local; restrict to affine

Recall notion of valuation ring

R int. domain, G value group, totally ordered

K quot. field of R

$$v : K - 0 \longrightarrow G \quad \text{s.t.}$$

$$v(xy) = v(x) + v(y)$$

$$v(x+y) \geq \min(v(x), v(y)) \quad \text{if } x+y \neq 0$$

$$R = \{ x \in K - 0 : v(x) \geq 0 \} \cup \{0\}$$

Claim R local, max ideal $m = \{x \in K : v(x) > 0\} \cup \{0\}$

Pf m ideal. Let $0 \neq a \in R$, $0 \neq b \in m$, show $ab \in m$

$v(ab) = v(a) + v(b) \geq 0 + 1 > 0$ so $ab \in m$. Similarly m closed under $+$

(R, m) local $\Longleftrightarrow$ $R - m = \{\text{units}\} \cup \{0\}$. But $0 \neq a \in R - m \Rightarrow v(a) \Rightarrow v(\frac{1}{a}) = 0$

$\Rightarrow 1/a \in R$, Easier to see a unit $\Rightarrow a \in R - m$ $\square$

$$\operatorname{Spec} R = \{ (0), m \}$$

$$\operatorname{Spec} K = \{ (0) \}$$

$t_0 = m$ closed

$t_1 = (0)$ generic