$\mathcal{F}, \mathcal{G}$ presheaves of abelian groups on X

Def a homomorphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

such that for each $V \subset U$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

An isomorphism is a homomorphism with a two-sided inverse

If $\mathcal{F}, \mathcal{G}$ are sheaves on X, a homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is a homom. of presheaves

Category $\mathcal{A}b(X)$

Local $\longleftrightarrow$ Global : Sheaf Theory [H II.1]

X topological space

$\mathcal{C}(X)$ ring of continuous fcns on X

$U \subset X$ open $\mathcal{C}(U)$

$U \subset V \, (\subset X)$ $f \in \mathcal{C}(V) \xrightarrow{\rho_{VU}} \mathcal{C}(U)$ restriction

$\rho_{UV}(f) = f|_V$

$U \subset V \subset W$ $f \in \mathcal{C}(W)$ $(f|_V)|_U = f|_U$

$\mathcal{C}(\emptyset) = 0$

$\mathcal{C}$ presheaf of rings on X

- $U = \bigcup U_i \qquad f \in \mathcal{C}(U) \qquad f|_{U_i} = 0 \quad \forall i$

$$\implies f = 0$$

- Given $f_i \in \mathcal{C}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \quad \forall i,j$

  $\exists f \in \mathcal{C}(U)$ s.t. $f|_{U_i} = f_i \quad \forall i$

$\mathcal{C}$ is a sheaf of rings on X

Let X be a topological space

Def A presheaf $\mathcal{F}$ of abelian groups (rings, sets, ...) on X is the assignment of an abelian group (ring, set, ...) $\mathcal{F}(U)$ to each open $U \subset X$ and to each inclusion of opens $V \subset U$ a homomorphism (morphism of rings, sets, ...)

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

such that

$\mathcal{F}(U) \to \mathcal{F}(W)$

- $\forall \; W \subset V \subset U, \quad \rho_{UW} = \rho_{VW} \circ \rho_{UV}$
- $\mathcal{F}(\emptyset) = 0$

Examples

- X algebraic variety $\mathcal{O}_X$ presheaf of regular functions on X, $\mathcal{O}_X(U) =$ ring of regular fcns on U

  $V \subset U$: $\rho_{UV} : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ is restriction

  $\mathcal{O}_X$ is a sheaf

- G abelian group, X top. space

  $\tilde{\mathcal{G}}$ constant presheaf

  $\tilde{\mathcal{G}}(U) = G \quad \forall U \neq \emptyset$

  $V \subset U \qquad \rho_{UV} = \mathrm{id}_G$

  $\tilde{\mathcal{G}}$ is <u>not</u> a sheaf

Often write $s|_V$ for $\rho_{UV}(s)$ for any sheaf

Ex $X = \mathbb{C}$ with Euclidean topology

$\mathcal{O}_{\mathbb{C},hol}$ = sheaf of holomorphic functions, +

$\mathcal{O}_{\mathbb{C},hol}(U)$ = " on U

$\mathcal{O}^{*}_{\mathbb{C},h}$ = sheaf of nowhere vanishing hol. fcns, $\times$

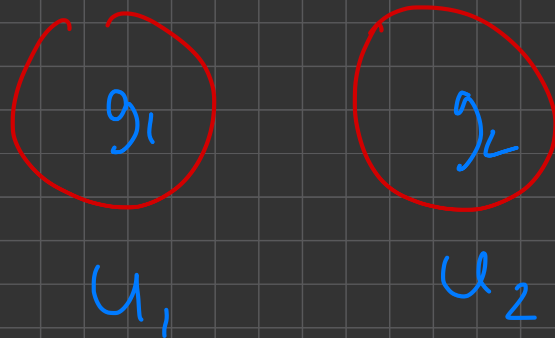
$$\varphi : \mathcal{O}_{\mathbb{C},hol} \longrightarrow \mathcal{O}^{*}_{\mathbb{C},hol}$$

$$\varphi_U \quad \mathcal{O}_{\mathbb{C},hol}(U) \longrightarrow \mathcal{O}^{*}_{\mathbb{C},hol}$$

$$f \longmapsto e^{2\pi i f}$$

Return to constant presheaf $\tilde{\mathcal{G}}$ associated to G

$\tilde{\mathcal{G}}(U_1 \cup U_2)$ $\qquad g_1 \neq g_2 \in G$



$\exists ?\ g \in G \qquad g|_{U_1} = g_1, \quad g|_{U_2} = g_2$

$U = U_1 \cup U_2$

However $\tilde{\mathcal{G}}$ has an associated sheaf $\mathcal{G}$

$\mathcal{G}(U)$ = set of locally constant functions $U \longrightarrow G$

$V \subset U$ ρ_{UV} restriction

(Agrees with description in [H])

Let $\mathcal{F}$ be a presheaf on X

Thm $\exists$ a sheaf $\mathcal{F}^+$ on X and a homomorphism $\varphi : \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal property. $\forall$ sheaves $\mathcal{G}$, homom $\mathcal{F} \to \mathcal{G}$ $\exists !$, hom $\mathcal{F}^+ \xrightarrow{\psi} \mathcal{G}$ s.t.

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\varphi"'] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, "\psi"'] & \end{tikzcd}$$

Furthermore $\mathcal{F}^+$ is unique, up to a unique isomorphism.

See [V 1.3] for generalities about universal properties

Please read the proof in the text!
It contains fundamental idea

Pf of uniqueness. Suppose $\mathfrak{Y}^+$, $\tilde{\mathfrak{Y}}^+$ both satisfy the universal property. We therefore have morphisms

$$\begin{tikzcd} & \mathfrak{Y}^+ \arrow[d, bend left, "\varphi"'] \\ \mathfrak{Y} \arrow[ur, "\alpha^+"] \arrow[r, "\tilde{\alpha}^+"'] & \tilde{\mathfrak{Y}}^+ \arrow[u, bend right=60, "\tilde{\varphi}"'] \end{tikzcd}$$

Since $\alpha^+ : \mathfrak{Y} \longrightarrow \mathfrak{Y}^+$ satisfies the universal property, have $\varphi : \mathfrak{Y}^+ \longrightarrow \tilde{\mathfrak{Y}}^+$, $\varphi \circ \alpha^+ = \tilde{\alpha}^+$

Similarly, have $\tilde{\varphi} : \tilde{\mathfrak{Y}}^+ \longrightarrow \mathfrak{Y}^+$, $\tilde{\varphi} \circ \tilde{\alpha}^+ = \alpha^+$

$$
\begin{tikzcd}
 & \mathcal{F}^+ \arrow[d, shift left, "\psi"] \\
\mathcal{F} \arrow[ur, "a"] \arrow[r, "\tilde{a}"'] & \tilde{\mathcal{F}}^+ \arrow[u, shift left, "\tilde{\psi}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[dr, "a"'] & \mathcal{F}^+ \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\tilde{\psi}\circ\psi"] \\
 & \mathcal{F}^+
\end{tikzcd}
$$

By uniqueness $\tilde{\psi}\circ\psi = \mathrm{id}_{\mathcal{F}^+}$

Similarly $\psi\circ\tilde{\psi} = \mathrm{id}_{\tilde{\mathcal{F}}^+}$

So ψ and $\tilde{\psi}$ are uniquely determined inverse isomorphisms

Language of limits in [H] is old. We will use more common modern terminology from [V]

[V]		[H]
limit	$\longleftrightarrow$	inverse limit
colimit	$\longleftrightarrow$	direct limit

We will discuss colimits and apply to the stalk $\mathcal{F}_p$ of a sheaf $\mathcal{F}$ at $p \in X$

Def: Let $\mathcal{D}$ be a small category (objects and morphisms are sets), $\mathcal{C}$ an arbitrary category. A diagram in $\mathcal{C}$ indexed by $\mathcal{D}$ is a functor

$$F : \mathcal{D} \longrightarrow \mathcal{C}$$

We write $C_i = F(i)$ for each $i \in Ob(\mathcal{D})$

A colimit of the diagram is an object denoted

$$\varinjlim_{\mathcal{D}} C_i \in Ob(\mathcal{C})$$

together with morphisms $f_i : C_i \longrightarrow \varinjlim_{\mathcal{D}} C_i$ $\forall i$ s.t.

$$\begin{tikzcd} C_i \arrow[dr, "f_i"] \arrow[d, "F(m)"'] & \\ C_j \arrow[r, "f_j"'] & \varinjlim_{\mathcal{D}} C_i \end{tikzcd}$$

commutes $\forall\ m : i \longrightarrow j$

which satisfies the following universal property:

$\forall \; B \in Ob(\mathcal{C})$ and morphisms $g_i : C_i \longrightarrow B$ $\forall i$ s.t.

$$\begin{tikzcd} C_i \arrow[d, "F(m)"'] \arrow[dr, "g_i"] & \\ C_j \arrow[r, "g_j"'] & B \end{tikzcd}$$

commutes $\forall \; m : i \to j$,

$\exists!$ $\varinjlim_{Q} C_i \longrightarrow B$ so that all of the following diagrams commute

$$\begin{tikzcd} C_i \arrow[d, "F(m)"'] \arrow[r, "f_i"] \arrow[rr, bend left, "g_i"] & \varinjlim_{Q} C_i \arrow[r, "\exists!"] & B \\ C_j \arrow[ur, "f_j"] \arrow[urr, bend right, "g_j"'] & & \end{tikzcd}$$

It follows that colimits, if they exist, are unique up to a unique isomorphism

Example: Cokernels

Let $\mathcal{D}$ be a category with two objects and two nontrivial morphisms $1 \overset{m}{\underset{n}{\rightrightarrows}} 2$.

Let $f: A_1 \to A_2$ be a homomorphism of abelian groups.

Let Ab be the category of abelian groups, and consider

$F: \mathcal{D} \to Ab$, $F(1) = A_1$, $F(2) = A_2$, $F(m) = f$, $F(n) = 0$

: $A_1 \overset{f}{\underset{0}{\rightrightarrows}} A_2$

Claim: $\varinjlim_{\mathcal{D}} A_i \cong \operatorname{coker} f$, together with morphisms

$A_1 \xrightarrow{0} \operatorname{coker} f$, $A_2 \to \operatorname{coker} f$ canonical map:

$A_1 \xrightarrow{g_1} B \xleftarrow{g_2} A_2$ s.t.

$$\begin{tikzcd} A_1 \arrow[d, "f"'] \arrow[r, "g_1"] \arrow[dr, phantom, "\circlearrowleft"] & B \\ A_2 \arrow[ur, "g_2"'] & \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[d, "0"'] \arrow[dr, "g_1"] & \\ A_2 \arrow[r, "g_2"'] & B \end{tikzcd} \text{ commute}$$

$\Longrightarrow$ $g_1 = 0$ $\Longrightarrow$ g_2 factors through $\operatorname{coker} f$

Stalks of Sheaves

$\mathcal{F}$ sheaf of Abelian grps on X, $p \in X$

$Ob(\mathcal{O}) = \{ p \in U \subset X \}$ open nbhd of p.

$\to Mor(U,V) = \{ V \subset U \}$

$F : \mathcal{O} \longrightarrow Ab$ $\qquad F(U) = \mathcal{F}(U)$

$F(V \subset U) = \rho_{UV} : F(U) \longrightarrow F(V)$

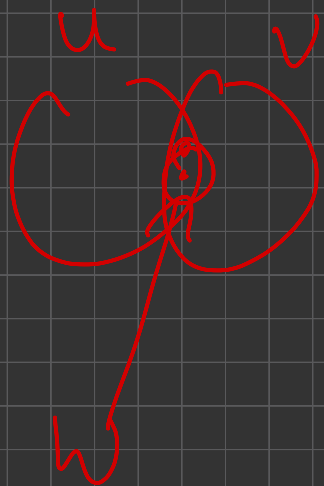
$\mathcal{F}_p := \varinjlim_{\mathcal{O}} \mathcal{F}(U)$, stalk of $\mathcal{F}$ at p.

Rk: via $\mathcal{F}(U) \longrightarrow \mathcal{F}_p$, $s \in \mathcal{F}(U)$ represents an elt of $\mathcal{F}_p$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "\rho_{UV}"'] & \mathcal{G} \\
\mathcal{F}(V) \arrow[ur] &
\end{tikzcd}$$

$V \subset U$

$$\mathcal{F}_p := \{\langle U, s\rangle \mid p\in U,\ s\in \mathcal{F}(U)\}/\sim$$



$$\langle U, s\rangle \sim \langle V, t\rangle \quad \text{if} \quad \exists\ p\in W\subset U\cap V$$

$$s|_W = t|_W$$

$\varphi : \mathcal{F} \to \mathcal{G}$ induces

$\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ $\quad \forall\, p\in X$

$$\begin{tikzcd}
 & \mathcal{F}_p \arrow[r, "\varphi_p"] & \mathcal{G}_p & \\
 \mathcal{F}(U) \arrow[ur] \arrow[r] \arrow[d, "\varphi(U)"'] & \mathcal{F}(V) \arrow[u] \arrow[d, "\varphi(V)"] & & \\
 \mathcal{G}(U) \arrow[r] \arrow[uurr] & \mathcal{G}(V) \arrow[uur] & &
\end{tikzcd}$$

$p\in V\subset U$

$$\begin{tikzcd}
 & \mathcal{G}_p & \\
 \mathcal{G}(U) \arrow[ur] \arrow[rr] & & \mathcal{G}(V) \arrow[ul]
\end{tikzcd}$$

Prop $\varphi : \mathcal{F} \to \mathcal{G}$ morph of sheaves on X

φ isomorphism $\Longleftrightarrow$ φ_p isom $\forall p$

Pf See text

$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves

Presheaf kernel $(U) = \ker(\mathcal{F}(U) \to \mathcal{G}(U))$

Presheaf image $(U) = \operatorname{Im}(\mathcal{F}(U) \to \mathcal{G}(U))$

Presheaf cokernel $(U) = \operatorname{coker}(\mathcal{F}(U) \to \mathcal{G}(U))$

$\ker \varphi$ = presheaf kernel

$\operatorname{im} \varphi = (\text{pre } \operatorname{Im})^{+}$

$\operatorname{coker} \varphi = (\text{pre } \operatorname{cok})^{+}$

Def φ injective if $\forall U$ $\varphi(U)$ injective

Important case: $\mathcal{F}' \subset \mathcal{F}$ subsheaf

Inclusion morphism $\iota : \mathcal{F}' \hookrightarrow \mathcal{F}$ injective.

$$\mathcal{F}/\mathcal{F}' := \operatorname{coker}(\iota)$$

Back to $\varphi : \mathcal{F} \to \mathcal{G}$ sheaves

$$\mathcal{F}(U) \longrightarrow \operatorname{im}(\varphi(U)) \subset \mathcal{G}(U)$$

$$\begin{tikzcd}
\mathcal{F} \arrow[r] & \text{Image presheaf} \arrow[r, hook] \arrow[d] & \mathcal{G} \\
& \operatorname{im}\varphi = (\text{image presheaf})^{+} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Claim $\operatorname{im}\varphi \longrightarrow \mathcal{G}$ injective

$\operatorname{im}\varphi$ "is" a subsheaf of $\mathcal{G}$

Def φ surjective if $\operatorname{im}\varphi = \mathcal{G}$

Ex $\mathcal{O}_{\mathbb{C},hol} \xrightarrow{\varphi} \mathcal{O}^{\times}_{\mathbb{C},hol}$ is surjective

$U = \mathbb{C}^{\times} = \mathbb{C} - 0$

$\mathcal{O}_{\mathbb{C},hol}(U) \longrightarrow \mathcal{O}^{\times}_{\mathbb{C},hol}(U)$ not surjective

$z \nearrow$

For any V, $f \in \mathcal{O}^{\times}_{\mathbb{C},hol}(V)$, show

$f \in (\operatorname{Im}\varphi)(V)$, i.e. $V = \bigcup V_i$

$f|_{V_i} \in$ (image presheaf) (V_i)

For this, take all V_i simply connected

$$f|_{V_i} = \varphi\left(\frac{1}{2\pi i}\log f|_{V_i}\right)$$

Exact sequence of sheaves

$$\dots \to \mathcal{F}^{i-1} \xrightarrow{\alpha^{i-1}} \mathcal{F}^{i} \xrightarrow{\alpha^{i}} \mathcal{F}^{i+1} \to \dots$$

exact $=$ $\ker \alpha^{i} = \operatorname{Im} \alpha^{i-1}$

$(\Leftrightarrow \forall p,\ \mathcal{F}_p^{i-1} \to \mathcal{F}_p^{i} \to \mathcal{F}_p^{i+1}$ exact$)$

$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G}$ f injection

$\mathcal{F} \xrightarrow{f} \mathcal{G} \to 0$ exact $\Leftrightarrow$ surj

$f: X \to Y$ continuous, $\mathcal{F}$ sheaf on X, $\mathcal{G}$ sheaf on Y

Def $f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ $\quad V \subset U \subset Y$, $f^{-1}(V) \subset f^{-1}(U)$

$$(f^{-1}\mathcal{G})(U) = \varinjlim_{f(U) \subset V \subset Y} \mathcal{G}(V) \quad U \subset X$$

$Z \overset{i}{\subset} X$ $\quad \mathcal{F}|_Z = i^{-1}\mathcal{F}$

Question: Let $\mathcal{B}$ be the sheaf of sections (to be described below) of a bundle B on X. Let $i: Z \hookrightarrow X$. Is $\mathcal{B}|_Z$ the sheaf of sections of the restriction of B to Z?

Answer: Yes

$B \xrightarrow{\pi} X$ bundle, typical fiber F

$\mathcal{B}$ sheaf (on X) of sections of B:

$U \subset X$ open

$$\mathcal{B}(U) = \left\{ s: U \to B \text{ s.t. } \begin{array}{l} a) \ s \text{ continuous} \\ b) \ \pi \circ s = \mathrm{Id}_U \end{array} \right\}$$